PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 70 JUNE, 1944 NO. 6

TECHNICAL PAPERS

AND

DISCUSSIONS

Published monthly, except July and August, at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York, N. Y. Reprints from this publication may be made on condition that the full title of Paper, name of Author, page reference, and date of publication by the Society, are given.

Entered as Second-Class Matter, September 23, 1937, at the Post Office at Lancaster, Pa., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 5, 1918.

Subscription (if entered before January 1) $8.00 per annum. Price $1.00 per copy

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Printed in the United States of America

		Discussion closes
Symposium: Dewatering, Incineration, and Use of Sewage Sludge Discussion in Feb., June, Sept., 1943, Feb. 1944	Jan., 1943	Closed*
Symposium: Rainbow Arch Bridge Over Niagara Gorge Discussion in Dec., 1943, Jan., Feb., Mar., Apr., May, June, 1944	Oct., 1943	Closed*
Sergev, S. Shear Effect on the Strength of Struts Discussion in Feb., Mar., 1944	Nov., 1943	Closed*
Clark, C. O. Storage and the Unit Hydrograph Discussion in Feb., Mar., May, June, 1944	Nov., 1943	July 1, 1944
Steinman, D. B. Rigidity and Aerodynamic Stability of Suspension Bridges Discussion in Feb., Mar., May, June, 1944	Nov., 1943	July 1, 1944
Symposium: Unusual Cutoff Problems—Dams of the Tennessee Valley Authority Discussion in Feb., Apr., May, 1944	Nov., 1943	July 1, 1944
Hunting, Marshall T. Geology in Highway Engineering Discussion in Feb., Mar., Apr., May, June, 1944	Dec., 1943	July 1, 1944
Elder, Clay C. Original and Readjusted Repayment Contracts, Boulder Canyon Project Discussion in Jan., Feb., 1944	Dec., 1943	July 1, 1944
Report: Failure of the Tacoma Narrows Bridge: Report of the Special Committee of the Board of Direction Discussion in Apr., June, 1944	Dec., 1943	July 1, 1944
Edwards, F. W., and Soucek, Edward. Surges in Panama Canal Reproduced in Model Discussion in Apr., May, June, 1944	Jan., 1944	July 1, 1944
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Symposium: Military Airfields Discussion in Feb., Mar., Apr., May, June, 1944	Jan., 1944	July 1, 1944
Wilson, David M. Analysis of Rigid Frames by Superposition Discussion in May, June, 1944	Feb., 1944	Aug. 1, 1944
Gray, Hamilton. Simultaneous Consolidation of Contiguous Layers of Unlike Compressible Soils Discussion in June, 1944	Feb., 1944	Aug. 1, 1944
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Benscoter, Stanley U., and Logan, Samuel T. Shear and Bond Stresses in Reinforced Concrete Discussion in May, June, 1944	Mar., 1944	Aug. 1, 1944
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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

TRANSPORTATION OF SUSPENDED SEDIMENT BY WATER

BY VITO A. VANONI,[1] ASSOC. M. AM. SOC. C. E.

SYNOPSIS

Measurements of the sediment and velocity distributions in a laboratory flume were made for various values of rate of flow, slope of channel, and size and amount of suspended load. The experimental sediment-distribution measurements were compared with the theoretical distribution derived on the assumption that the coefficient for the turbulent transfer of suspended sediment is the same as the coefficient for the turbulent transfer of momentum.

The following are the main findings:

(1) The measured sediment distributions have the same form as the theoretical distribution but do not agree quantitatively with them. This shows how invalid is the basic assumption that the transfer coefficient for sediment was equal to that for momentum. In general, the measured relative concentrations were larger than those given by the theory, the difference being greatest for the fine sediment and diminishing as the suspended load became coarser.

(2) Suspended sediment in a flow tends to reduce appreciably the turbulent transfer of momentum and hence the resistance to flow, allowing the sediment-laden water to flow more rapidly than a comparable clear water flow. This effect varies with the total sediment load and the grain size of the sediment.

(3) Suspended load in a flow may be a factor in causing the secondary circulation observed in rivers.

INTRODUCTION

Many important practical problems of today involve the movement of sediments by water, and their complete solution requires more knowledge of

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by November 1, 1944.

[1] Project Supervisor, Cooperative Laboratory, U. S. Dept. of Agriculture, SCS, Asst. Prof. of Hydraulics, California Inst. of Technology, Pasadena, Calif.

this subject than now exists. Flood control and channel maintenance on streams, both large and small, are just as much problems of sediment movement as of water flow. Established practices, in fact, are successful in handling the water flow of floods, but the problem of passing the sediment load is still unsolved. In recent years the work of the United States Department of Agriculture in developing methods of reducing accelerated erosion on agricultural lands has encountered many new sediment-transportation problems, which have shown the urgent need for additional fundamental knowledge of this subject. It is in connection with this need that this study was undertaken.

Sediment may be transported by flowing water in two essentially different ways—by rolling or sliding along the bed of the stream channel or in suspension in the body of the fluid. The material transported is called, respectively, the "bed load of the stream," or simply "bed load," and the "suspended load." Just above the bed a continual interchange of material is occurring between the bed and the overlying fluid and in this locality it is obviously difficult to distinguish between the bed load and suspended load. The two types of transportation are by no means independent. They are separated only for convenience in studying them and referring to them.[2] This paper deals only with suspended load.

Previous Work

Theories.—Modern theories of suspended-load transportation are based on developments in the mechanics of turbulent flow of fluids. Consider a two-dimensional turbulent flow in the x-direction with a velocity $u(y)$, in which y is normal to x. Then, according to Osborne Reynolds,[3] the shear parallel to x on a plane normal to y may be expressed by

$$\tau = -\rho \overline{u' v'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

in which ρ is the mass density of the fluid, and u' and v' are the turbulent velocity fluctuations in the x and y directions, respectively, and the bar denotes mean value.

M. J. Boussinesq[4] introduced an expression for the shear of the form,

$$\tau = \rho \epsilon_m \frac{du}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

in which the quantity $\rho \epsilon_m$ (often called the eddy viscosity) is analogous to the coefficient of viscosity in the expression for viscous shear; and u is the mean velocity in the x-direction. The quantity ϵ_m has the dimensions of a kinematic viscosity ν, but unlike its counterpart it is a function of flow conditions and, therefore, varies from point to point. The quantity ϵ_m is a coefficient expressing the exchange of momentum between neighboring filaments.

[2] "A Distinction Between Bed-Load and Suspended Load in Natural Streams," by H. A. Einstein, A. G. Anderson, and J. W. Johnson, *Transactions*, Am. Geophysical Union, Pt. II, 1940, pp. 628–633.

[3] "On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion," by Osborne Reynolds, *Catalogue of Scientific Papers (1800–1900)*, compiled by the Royal Soc. of London, Cambridge Univ. Press, Vol. 2, pp. 535–577; also *Philosophical Transactions*, Royal Soc. of London, Series A, Vol. 186, 1895, p. 123.

[4] "Essai sur la Theorie des Eaux Courantes," by M. J. Boussinesq, *Mémoires, Presents par divers Savants*, L'Académie de l'Institut de France, Vol. 23, 1877, pp. 1–680.

This quantity is called the turbulent momentum transfer coefficient. Obviously it has the dimensions, length × velocity.

Theodor von Kármán,[5] M. Am. Soc. C. E., introduced the equation,

$$\tau = \beta \rho v' l \frac{du}{dy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which β is a correlation coefficient, v' is the average of the absolute values of the fluctuations normal to the main flow, and l is the mixing length. From Eq. 2 it is seen that the turbulent momentum transfer coefficient, ϵ_m, has the value,

$$\epsilon_m = \beta v' l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and the dimensions, velocity × length.

Just as Eq. 2 expresses the rate of transfer of momentum due to turbulent mixing, it is possible to write by a similar equation the transfer of suspended particles, or any other property that is carried by the fluid. Thus, the rate of transfer of mass of suspended particles per unit area is

$$G = -\beta v' l \frac{dC}{dy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which C is the concentration of suspended material in mass per unit volume. Professor von Kármán[5] pointed out that the values of β, v', and l in Eqs. 2, 3, and 4 need not necessarily be the same.

For steady conditions the upward transport of material due to turbulence is balanced by the settling due to the force of gravity. This is expressed by

$$-\beta v' l \frac{dC}{dy} = w C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which C is now the concentration of the fraction of the sediment with a settling velocity w. Transposing, the differential equation for suspended sediment becomes,

$$w C + \epsilon_s \frac{dC}{dy} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which ϵ_s is the sediment transfer coefficient that is similar to ϵ_m but not necessarily equal to it. Solution of Eq. 6 gives,

$$\log_e \frac{C}{C_a} = -w \int_a^y \frac{dy}{\epsilon_s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

in which C_a is the concentration at an arbitrary reference level $y = a$.

If ϵ_s is constant over the depth, that is, $\epsilon_s(y)$ = constant, Eq. 7 may be integrated to give

$$\frac{C}{C_a} = e^{-i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

[5] "Some Aspects of the Turbulence Problem," by Theodor von Kármán, *Proceedings*, 4th International Cong. of Applied Mechanics, 1934, pp. 54–91.

in which e is the base of the natural logarithms; and, to simplify typography,

$$i = \frac{w}{\epsilon_s}(y - a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Eq. 8 was apparently first given by Wilhelm Schmidt,[6] who applied it to the study of suspension of dust particles in the atmosphere. Eq. 6 was derived by a formal procedure by Morrough P. O'Brien,[7] M. Am. Soc. C. E. The more direct derivation is the work of Professor von Kármán.[5]

In general, ϵ_s is a function of y and must be so expressed before Eq. 7 can be integrated. Since the expression for ϵ_s is not known the assumption is made that $\epsilon_s = \epsilon_m$ and Eq. 7 is integrated on this basis. From Eqs. 1*b* and 2,

$$\epsilon_m = \beta\, v'\, l = \frac{\tau}{\rho \dfrac{du}{dy}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Introducing Eq. 10 into Eq. 7 yields the relation

$$\log_e \frac{C}{C_a} = -\rho\, w \int_a^y \frac{\dfrac{du}{dy}}{\tau}\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

which was first presented by Professor von Kármán[5] in 1934.

In a uniform open-channel flow whose ratio of width to depth is large, these relations will apply. The expression for the shear stress is given by

$$\tau = \tau_0 \frac{y_m - y}{y_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

in which τ_0 is the shear at the bottom, or,

$$\tau_0 = \gamma\, y_m\, S \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12b)$$

Introducing Eq. 12a into Eq. 11,

$$\log_e \frac{C}{C_a} = \frac{w}{\dfrac{\tau_0}{\rho}} \int_a^y \frac{\dfrac{du}{dy}}{\dfrac{y_m - y}{y_m}}\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

An expression for the velocity gradient is given by the von Kármán[8] universal velocity defect law,

$$\frac{u - u_{\max}}{\sqrt{\dfrac{\tau_0}{\rho}}} = \frac{1}{k} \log_e \frac{y}{r_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

---

[6] "Der Massenaustauch in freier Luft und verwandte Erscheinungen," by Wilhelm Schmidt, *Probleme der kosmischen Physik*, 1925, Pt. 7.

[7] "Review of the Theory of Turbulent Flow and Its Relation to Sediment-Transportation," by M. P. O'Brien, *Transactions*, Am. Geophysical Union, Section of Hydrology, 1933, pp. 487–491.

[8] "Turbulence and Skin Friction," by Theodor von Kármán, *Journal of the Aeronautical Sciences*, Vol. 1, No. 1, 1934 pp. 1–20.

which has been shown by J. Nikuradse[9,10] to apply to pipes and by G. H. Keulegan[11] to apply also to open channels. In Eq. 14, k is a universal constant (0.40 for pipes) and y is the distance from the wall of the pipe of radius r_0. For wide open channels, as discussed here, r_0 in Eq. 14 is replaced with y_m, the depth of the flow. The derivative of the velocity then becomes,

$$\frac{du}{dy} = \frac{1}{k}\sqrt{\frac{\tau_0}{\rho}}\,\frac{1}{y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

which, when introduced into Eq. 13, gives

$$\log \frac{C}{C_a} = -\frac{w}{k\sqrt{\frac{\tau_0}{\rho}}}\int_a^y \frac{dy}{y\left(\frac{y_m - y}{y_m}\right)} \ldots\ldots\ldots\ldots\ldots\ldots (15b)$$

Integrating Eq. 15*b*,

$$\frac{C}{C_a} = \left(\frac{y_m - y}{y} \times \frac{a}{y_m - a}\right)^z = H^z \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which, to simplify typography,

$$z = \frac{w}{k\sqrt{\frac{\tau_0}{\rho}}} = \frac{w}{k\sqrt{g\,y_m\,S}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

From Eq. 16 it is possible to plot curves of relative sediment distribution as was first done about 1936 by Arthur T. Ippen, Assoc. M. Am. Soc. C. E., at the suggestion of Professor von Kármán.[12]

Eq. 16, as well as the equations preceding it, gives only a relative concentration and therefore does not make possible the calculation of the total transport of the stream. To make this possible a value for C_a is needed, and the problem of obtaining the transporting capacity of a flow may be simply stated as one of determining this value. In discussing the magnitude of C_a, Professor von Kármán[5] states that it probably depends on the size of the sediment and on the magnitude of the shearing stress acting at the ground. Hunter Rouse[13] has suggested that the limiting value of a in Eq. 16 be made equal to the height of the roughness elements k_s. He further suggests that the corresponding concentration C_k for this level be calculated from Eq. 8, assuming that in the interval $y = 0$ to $y = k_s$, $\epsilon(y) = \epsilon(k_s) =$ constant, and that the concentration at $y = 0$ be set equal to that of the bed material. E. W. Lane, M. Am. Soc. C. E., and A. A. Kalinske, Assoc. M. Am. Soc. C. E.,

[9] "Gesetzmässigkeiten der turbulenten Strömung in Glatten Rohren," by J. Nikuradse, *Forschungsheft 356*, Vereines Deutscher Ingenieure, Vol. 3, 1932, p. 36.

[10] "Strömungsgesetze in Rauhen Rohren," by J. Nikuradse, *Forschungsheft 361*, Vereines Deutscher Ingenieure, Vol. 4, 1933, pp. 1–22.

[11] "Laws of Turbulent Flow in Open Channels," by Garbis H. Keulegan, *Journal of Research*, National Bureau of Standards, U. S. Dept. of Commerce, Vol. 21, 1938, pp. 707–741.

[12] "Modern Conceptions of the Mechanics of Fluid Turbulence," by Hunter Rouse, *Transactions*, Am. Soc. C. E., Vol. 102 (1937), Fig. 30, p. 536.

[13] "An Analysis of Sediment Transportation in the Light of Fluid Turbulence," by Hunter Rouse, *Mimeograph Publication SCS-TP-25*, U.S.D.A., 1939.

approaching the problem statistically, argue[14] that only those particles having settling velocities w less than the vertical fluctuations v' of the bed velocity will be picked up. By assuming a normal distribution of v' and setting the root mean-square fluctuation $\sqrt{(v')^2}$ proportional to the friction velocity $\sqrt{\frac{\tau_0}{\rho}}$ they proceed to calculate the probable rate of suspension of material.

As a result of an analysis of the energy balance of a stream carrying suspended load, Robert T. Knapp,[15] M. Am. Soc. C. E., advanced the hypothesis that the capacity of a stream is limited only if the settling velocity w of the sediment is greater than the rate of fall of the stream $\bar{u}\,S$, in which $\bar{u}$ is the average velocity of flow and S is the slope of the channel. Furthermore, he deduced that when the settling velocity is less than $\bar{u}\,S$ the material contributes energy to the flow, thus making it possible to pick up additional load until extremely high concentrations are reached.

Verification of Theories.—Although no comprehensive suspended-load experiments have been made in which the important variables have been investigated systematically, some noteworthy experimental work has been done.

Mr. Schmidt[6] appears to have first applied the turbulence theory to the problem of suspended load in his studies of the suspension of dust particles in air. He obtained Eq. 8, which assumes a constant value for the exchange coefficient.

In 1929 H. E. Hurst,[16] Jun. Am. Soc. C. E., reported a series of experiments on the suspension of sediment in a cylindrical column of water agitated uniformly by a series of propellers much the same as those used by Joule in his experiments on the mechanical equivalent of heat. At various levels in his apparatus he measured the concentrations of three well-graded sands (mean size 0.9 mm, 0.4 mm, and 0.2 mm) for several propeller speeds and found that the distribution of sediment followed an exponential relationship of the form of Eq. 8. Mr. Hurst was also able to derive his relationship from an analysis based on the kinetic theory of gases.

Professor Rouse[17] performed some experiments similar to those by Mr. Hurst using an agitator of coarse mesh bars that was oscillated in the vertical direction with simple harmonic motion. The results of Professor Rouse's work indicate that Eq. 8 is valid, and verify the theory. Professor Rouse observed a systematic deviation of the value of ϵ obtained for the coarse sand from that obtained from fine sand. The deviation was attributed to a difference in the correlation factor β of Eq. 3 for mixing of the fluid and of the sediment.

In 1932 John B. Leighly[18] outlined a method of calculating the exchange coefficient ϵ for natural streams and discussed its effect in suspending sediment

[14] "The Relation of Suspended to Bed Material in Rivers," by A. A. Kalinske and E. W. Lane, *Transactions*, Am. Geophysical Union, Pt. 4, 1939, pp. 637–641.

[15] "Energy-Balance in Stream-Flows Carrying Suspended Load," by Robert T. Knapp, *ibid.*, Pt. 1, 1938, pp. 501–505.

[16] "The Suspension of Sand in Water," by H. E. Hurst, *Proceedings*, Royal Soc. of London, Series A, Vol. 124, 1929, pp. 196–201.

[17] "Experiments on the Mechanics of Sediment Suspension," by Hunter Rouse, *Proceedings*, 5th International Cong. for Applied Mechanics, 1938, pp. 550–554.

[18] "Toward a Theory of the Morphologic Significance of Turbulence in the Flow of Water in Streams," by John B. Leighly, *University of California Publications in Geography*, Vol. 6, No. 1, 1932, pp. 1–22.

in the body of the flow. In a later paper Mr. Leighly[19] calculated the exchange coefficient from some sediment measurements on natural streams by using Eq. 8. He also calculated the sediment distribution from the exchange coefficient by step integration.

In 1935 J. E. Christiansen,[20] Assoc. M. Am. Soc. C. E., attempted to check Eq. 7 by applying it to sediment measurements made by Harry F. Blaney and the late Samuel Fortier, Members, Am. Soc. C. E., in the Colorado River and Imperial Canals.[21] In general the observations agreed with Eq. 7, but the results were not entirely conclusive since complete data were not available and it was necessary to estimate such quantities as the slope of the canals and the settling velocity of the sediment before calculations could be made.

E. G. Richardson[22] made some simultaneous sediment and velocity measurements in a flume 5 cm wide, 8 cm deep, and 150 cm long which led him to conclude that the vertical distribution of sediment is nearly exponential. This agrees with Eq. 8 and would indicate that the mixing coefficient had approximately a constant value over the depth. Such a condition could be approximated in a channel downstream from a disturbance such as might occur in the inlet to a flume. In a later paper[23] this same author reports the results of experiments in a somewhat larger channel (1 ft by 1 ft by 6 ft) and concludes that in the boundary region the concentration of suspended sediment is inversely proportional to y and that in the open stream it varies exponentially, much as expressed by Eq. 8.

Apparatus and Procedure

Main Flume.—The experiments were performed in a flume 33¼ in. wide by 60 ft long, arranged so the slope and discharge could be varied. The slope was adjusted by means of four pairs of screw jacks, and the discharge was varied from 1 cu ft per sec to about 5.5 cu ft per sec by changing the speed of the pump in the system through a variable-ratio transmission in the electric motor drive. The sides of the channel were made of 12-in. structural channels lined with ⅛-in. rubber. The bottom was a $\frac{5}{16}$-in. steel plate artificially roughened with sand. By making the circulation system closed circuit with the minimum velocity in the flume, it was possible to recirculate the suspended sediment with the water without danger of depositing material in the piping system.

The flume as first constructed was described in considerable detail[24] by Robert T. Knapp and Robert M. Oaks and is shown diagrammatically in Fig. 1(*a*). Experiments with this circuit yielded fair results but indicated that

[19] "Turbulence and the Transportation of Rock Debris by Streams," by John B. Leighly, *The Geographical Review*, Vol. 24, No. 3, 1934, pp. 453–464.

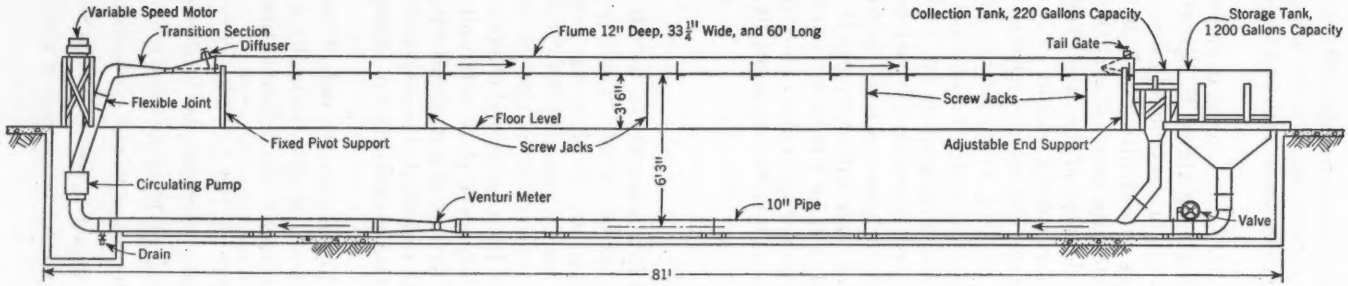
[20] "Distribution of Silt in Open Channels," by J. E. Christiansen, *Transactions*, Am. Geophysical Union, Pt. 2, 1935, pp. 478–485.

[21] "Silt in the Colorado River and Its Relation to Irrigation," by Harry F. Blaney and Samuel Fortier, *Technical Bulletin No. 67*, U.S.D.A., 1928.

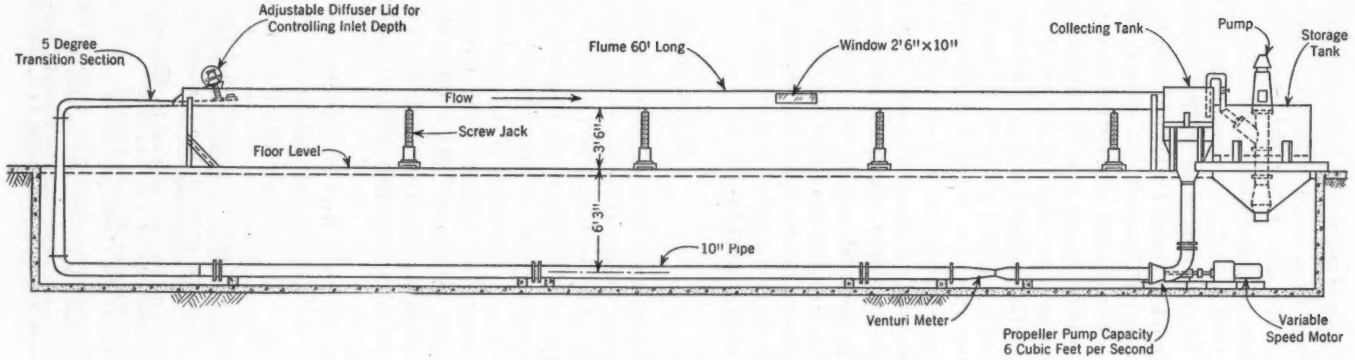
[22] "The Transport of Silt by a Stream," by E. G. Richardson, *Philosophical Magazine and Journal of Science*, Vol. 17, No. 7, 1934, pp. 769–783.

[23] "The Suspension of Solids in a Turbulent Stream," by E. G. Richardson, *Proceedings*, Royal Soc. of London, Series A, Vol. 162, 1937, pp. 583–597.

[24] "A 'Closed-Circuit' Flume for Suspended-Load Studies," by Robert M. Oaks and Robert T. Knapp, *Transactions*, Am. Geophysical Union, Pt. 2, 1936, pp. 525–528.



(*a*) FIRST EXPERIMENTS, SERIES 1, RUNS 1 TO 13



(*b*) FINAL EXPERIMENTS, SERIES 2, RUNS 14 TO 22

FIG. 1.—FLUME CIRCUITS USED FOR EXPERIMENTS

the disturbances due to the pump and inlet were exerting an appreciable effect on flow in the flume.

The flow was improved by revising the circuit as shown in Figs. 1(*b*) and 2. Moving the pump and venturi meter to the discharge end of the flume provided a run of about 50 ft of 10-in. pipe to "equalize" the flow before it entered the transition to the 33¼-in. flume section. To eliminate the possibility of separation the included angle between the sides of the transition was made 5°.

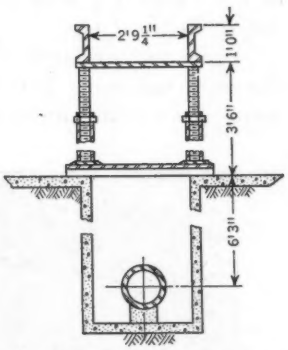


FIG. 2.—SECTION THROUGH A-A, FIG. 1(*b*)

Fig. 3 shows a general view of the flume from the inlet end. The beam spanning the channel at about midlength carries the instruments used in making measurements. The diffuser lid at the flume inlet is shown at the left.

Adjustment of Flow.—The flow in the flume was readily adjusted to a given depth by pumping the correct volume of water into the system and varying the discharge until uniform flow was reached. Uniform flow for a given discharge was obtained easily (once the flow was set with the water manometer on the venturi meter) by adjusting the amount of water in the closed system. No difficulty was ever experienced with the closed system because of pendulation or any other instability. The flow was adjusted easily and remained steady for indefinite lengths of time.

Depth Measurements.—Water depths were measured with a point gage which can be read to 0.001 ft by means of a vernier scale. Depth measurements were made by first adjusting the elevation of the bracket holding the instrument so that the scale read zero when the point was resting on the bottom and then lifting the point so that it just came in contact with the surface. The average depth for the channel was obtained from measurements of at least 10 or 15 points.

Velocity Measurements.—Velocities were measured with a pitot static tube (diameter, $\frac{3}{16}$ in.) of the standard Prandtl design. The differential pressure on the tube was read to 0.001 ft on a water manometer and converted to velocity by assuming a coefficient of unity for the tube. (That this assumption is justified is shown by H. Peters.[25])

Sediment Sampler.—The distribution of sediment was determined from samples siphoned from the flume through a brass-tube pipette (outside diameter, $\frac{5}{16}$ in.) shaped much like a pitot tube. The tip of the sampler was flattened so its inside dimensions were 0.490 in. by 0.139 in. The average velocity at which the sediment-laden fluid entered the flat-tip section (sampling velocity) was made equal to the stream velocity at the sampling point by adjusting the head on the siphon.

The time to remove a 1-liter sample was calculated for each point from the measured stream velocity and the area of the flat-tip sampler (0.064 sq in.),

[25] "Druckmessung," by H. Peters, *Handbuch der Experimentalphysik*, Leipzig, Vol. 4, Pt. 1, 1931, pp. 489–510.

the head on the siphon required for the rate of flow was determined by trial, and three 1-liter samples were taken. The sediment was removed from the sample by filtering and was dried and weighed. The average weight of the three samples was taken as the concentration at the point. Sample weights of more than 1 g were determined to 1 cg, and the smaller quantities were determined to 1 mg.

Water removed from the flume in sampling was replaced in order to keep the depth uniform. It was found that removing or adding as little as 3 gal of water made a measurable difference in the tailwater depth.

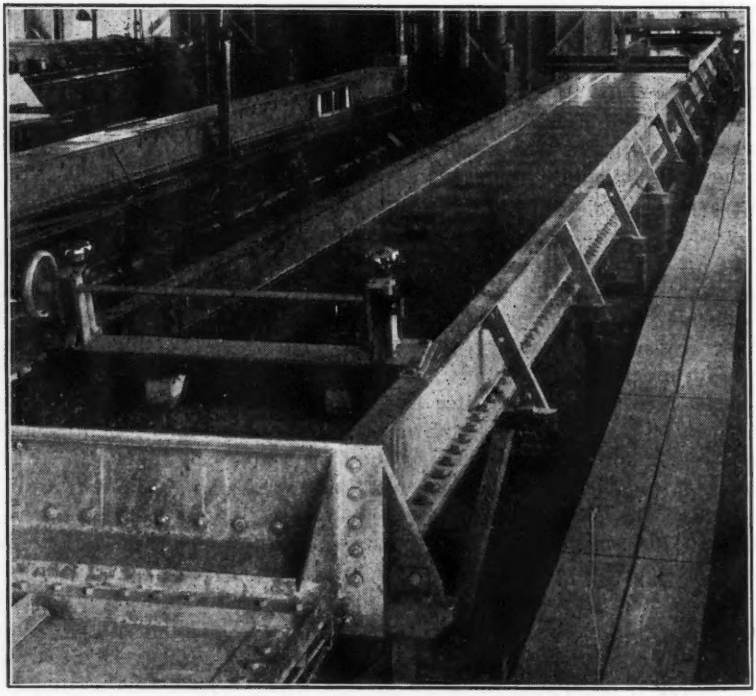


FIG. 3.—GENERAL VIEW OF FLUME FROM INLET END

The effect of the sampling velocity on the amount of sediment removed with the sample was studied experimentally. Several 1-liter samples were removed from a point at sampling velocities equal to 1.00, 0.86, and 0.75 times the local stream velocity at the sampling point. The sampling point was 0.5 ft from the center of the flume, 44 ft from the inlet (that is, at station 44), and 0.1 ft from the bottom. The local velocity was 3.19 ft per sec, the flume was set to a slope of 0.0025, and the water was flowing at an equilibrium depth of 0.495 ft. The suspended load had a mean settling rate in still water of 0.058 ft per sec.

The results of this experiment, presented in Table 1, show that, as the sampling velocity decreases, the quantity of sediment collected per given volume of fluid increases. When the sampling velocity is lower than the local stream velocity, the fluid stream lines curve away from the sampler tip.

TABLE 1.—EXPERIMENTAL DETERMINATION OF THE EFFECT OF THE SAMPLING VELOCITY ON THE AMOUNT OF SEDIMENT COLLECTED

No.	Observations[a]	$\frac{u_s}{u} = 1.00$	$\frac{u_s}{u} = 0.86$	$\frac{u_s}{u} = 0.75$	Definitions
1	$\overline{C}$	1.03	1.08	1.12	[a] u_s/u is the ratio of sampling velocity to the velocity of the stream at the sampling point; $\overline{C}$ is the average sediment concentration, in grams per liter, for the series of samples; $\overline{C}/\overline{C}_s$ is the ratio of $\overline{C}$ to the average concentration obtained when the sampling velocity is equal to the stream velocity; and $C/\overline{C}$ is the ratio of the sediment concentration for individual sample to $\overline{C}$.
2	$\frac{\overline{C}}{\overline{C}_s}$	1.00	1.05	1.09	
3	$\frac{C}{\overline{C}}$	0.99	0.98	1.13	
		1.04	0.94	0.99	
		0.99	1.08	0.95	
		0.95	0.90	0.99	
		0.97	1.00	1.00	
		1.06	0.99	0.97	
		....	1.05	....	
		....	0.91	....	
		....	1.10	....	

Because of their greater inertia the sediment particles follow paths of less curvature than the fluid, with the result that sediment is transferred to the fluid entering the sampler and the apparent concentration is increased. When the sampling velocity is greater than the stream velocity the curvature is toward the sampler, and a tendency to decrease the amount of sediment collected would therefore be expected.

The values of $C/\overline{C}$ in line 3, Table 1, give some idea of the fluctuation in concentration at a point in the flow. Although the deviation of the weights of individual samples from the mean was as high as 10%, it is interesting to note that the deviation of the average weight of three consecutive samples from the mean does not exceed 3%. Furthermore, the length of a filament of fluid having a volume of 1 liter and an area equal to that of the sampler tip is well in excess of the length of the flume, which is 60 ft.

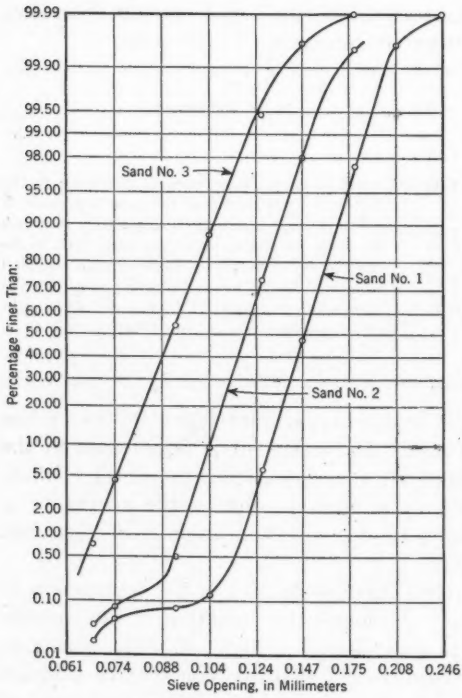
Sediment Used as Suspended Load.—Sediments used as suspended load were prepared by grading a silica sand in a wind-tunnel classifier constructed[26] by George H. Otto and Professor Rouse. Fig. 4(*a*) shows a logarithmic probability chart of the size distributions of the three sands used as suspended load. The parameters that describe size and grading of these sands are listed in Fig. 4(*b*).

The quantity D_m is the geometric mean sieve diameter and is read directly from the chart. For a normal distribution (that is, when the curve is a straight line), the geometric mean and median diameters are equal. The quantity σ_g (Fig. 4(*b*)) is known as the geometric standard deviation of the sieve diameters. It is the ratio of the geometric mean size D_m to the size corresponding to the ordinate value 15.9%, which in turn is also equal to the ratio of the size corresponding to the ordinate value 84.1% to D_m. The method of plotting the

[26] "Wind-Tunnel Classifier for Sand and Silt," by George H. Otto and Hunter Rouse, *Civil Engineering*, July, 1939, pp. 414–415.

size distribution and of calculating these parameters was shown by Mr. Otto.[27]

The quantity $\bar{D}_s$ is a weighted mean sedimentation diameter, the sedimentation diameter of a particle being defined as the diameter of a sphere having the same density and the same settling velocity as the particle. The mean settling velocity $\bar{w}$ of the center of mass of a group of particles was determined from observed settling times of individual particles of a representative sample of the sediment. The mean sedimentation diameter $\bar{D}_s$ is then the diameter of the quartz sphere having the settling velocity $\bar{w}$ in water. Calculation of the sedimentation diameter was based on the resistance coefficients for spheres as outlined by Professor Rouse.[28] The mean sedimentation diameter was also determined from sieve analyses by taking a weighted mean of the sedimentation diameters of the material held in each sieve, which were determined from actual settling-velocity measurements of samples taken from the sieve separates.



(*a*) SIZE DISTRIBUTION

(*b*) SEDIMENTATION DIAMETERS IN MILLIMETERS

Sand No.	Geometric mean sieve size D_m	Geometric standard deviation σ_g	Mean sediment diameter $\bar{D}_s$
1	0.147	1.10	0.160
2	0.120	1.11	0.133
3	0.091	1.13	0.100

FIG. 4.—LOGARITHMIC PROBABILITY GRAPH OF SIZE DISTRIBUTION OF SANDS USED AS SUSPENDED LOAD

The average sediment concentration C_m for the entire flow was determined from samples siphoned from the collection tank through a ¾-in. pipe and rubber hose. The entrance of the siphon was directed vertically upward so the downward flow in the tank could enter without changing direction. During sampling the pipe was moved around to obtain fluid from all parts of the cross section.

Bottom Roughness.—As mentioned previously, the sides of the flume were lined with rubber, and the bottom was roughened with sand. The two bottom

[27] "A Modified Logarithmic Probability Graph for the Interpretation of Mechanical Analyses of Sediments," by George H. Otto, *Journal of Sedimentary Petrology*, Vol. 9, No. 2, 1939, pp. 62–76.

[28] "Nomogram for the Settling Velocity of Spheres," by Hunter Rouse, Exhibit D of the Annual Report of the Committee on Sedimentation, National Research Council, Div. of Geology and Geography, Washington, D. C., 1936–1937, pp. 57–64.

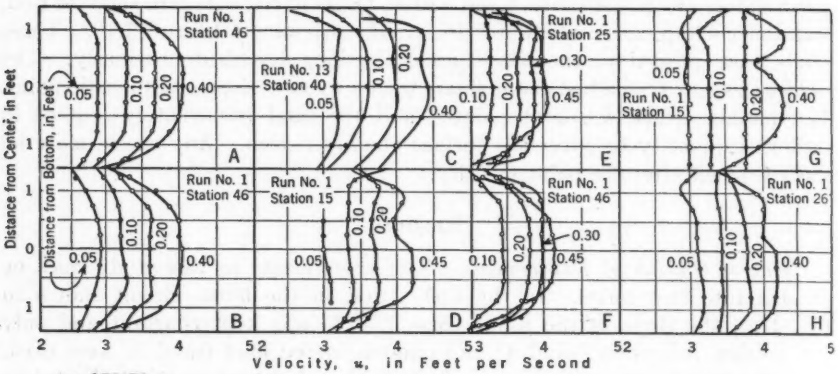
roughnesses used in the experiments were designated as sands 4 and 5. The mean diameters $\bar{D}_s$ for sands 4 and 5 were, respectively, 0.47 mm and 0.88 mm, and the corresponding standard deviations, σ_g, were 1.12 and 1.10. These sands were graded in the wind-tunnel classifier described previously. The sand was stuck to the bed of the flume with a water-resistant bitumastic paint. The metal surface was first painted and the sand immediately applied in sufficient quantity to cover the surface one layer deep. Any excess material was swept off after the paint was dry.

Results

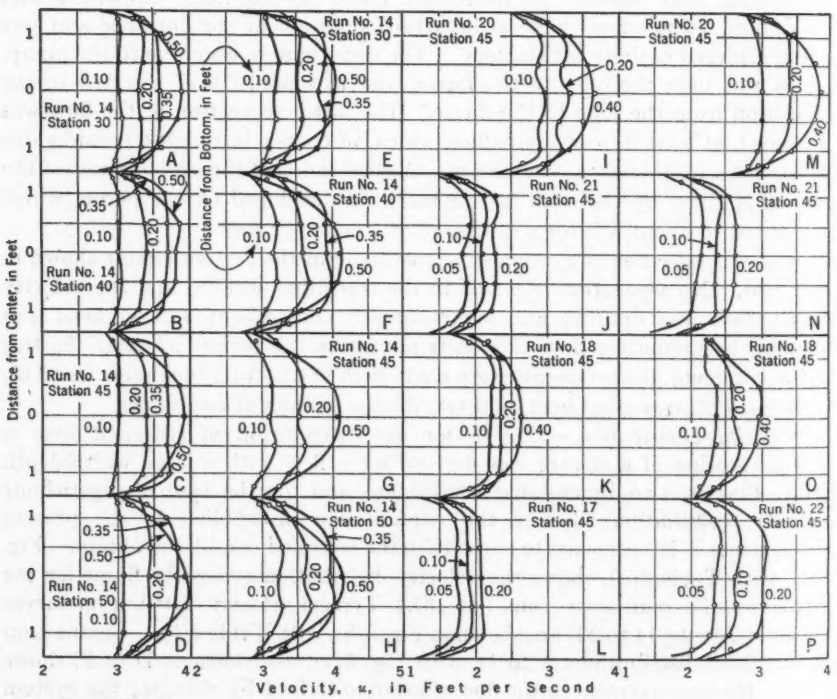
General Outline of Experiments.—The experiments in this study may be divided into two series. In the first series, in the flume circuit shown in Fig. 1(*a*), the slope of the flume ($S = 0.0025$) was kept constant and only one bottom roughness (sand 4) and one suspended load (sand 1) were used. Only the rate at which water was discharged and the amount of sediment in the system were varied. As mentioned under the heading, "Apparatus and Procedure," this circuit gave a disturbed condition at the entrance and was not considered entirely satisfactory. The disturbances were due to the pump, which was near the inlet to the flume, and to separation of the flow in the transition from the pipe to the flume. The latter separation of the flow was remedied, at least in part, by baffles, which, of course, introduced considerable resistance and, therefore, turbulence. A good index of the performance of the system is given by the head loss or the head developed by the pump, which amounted to about 5 ft for a discharge of 5 cu ft per sec.

The second series of experiments was made in the revised circuit shown in Fig. 1(*b*). No separation occurred in the transition section, and the velocity distribution was uniform and symmetrical. Only about 3 ft of head was required to circulate a flow of 5 cu ft per sec in the revised system. In this series, complete measurements were made with one bottom roughness (sand 5), three sizes of suspended load, and two depths at each of two slopes.

Velocity Distribution.—Eq. 16 for the distribution of sediment over a vertical section of a stream was derived for a flow with a large width-depth ratio—that is, two-dimensional conditions—and for the case of logarithmic velocity distribution. Before the relation can be applied to the present experiments it is necessary to show that the assumed conditions obtain. Fig. 5(*a*), with Table 2(*a*), shows the velocity distribution across the flume for the first series of experiments (runs 1 to 13). Typical velocity distribution curves for series 2 (runs 14 to 22) are shown in Fig. 5(*b*), with Table 2(*b*). Comparison of Fig. 5(*b*), distributions A to D, with Fig. 5(*a*), distributions D to F, shows at once the improvement in the flow effected in series 2 by changing the system from that shown in Fig. 1(*a*) to the one in Fig. 1(*b*). These velocity distributions also show that the influence of the walls of the channel does not extend to the center of the flume and that the assumption of two-dimensional conditions near the center of the flume is justified. The only way the resistance of the walls can be transferred to the center of the channel is by turbulent shear or momentum transfer, which can exist only if there is a horizontal velocity


Run No. 1 Station 46
Run No. 13 Station 40
Run No. 1 Station 25
Run No. 1 Station 15
Run No. 1 Station 46
Run No. 1 Station 15
Run No. 1 Station 46
Run No. 1 Station 26
Distance from Center, in Feet
Distance from Bottom, in Feet
Velocity, u, in Feet per Second

(*a*) SERIES 1


Run No. 14 Station 30
Run No. 20 Station 45
Run No. 20 Station 45
Run No. 14 Station 40
Run No. 21 Station 45
Run No. 21 Station 45
Run No. 14 Station 45
Run No. 18 Station 45
Run No. 18 Station 45
Run No. 14 Station 50
Run No. 17 Station 45
Run No. 22 Station 45
Distance from Center, in Feet
Distance from Bottom, in Feet
Velocity, u, in Feet per Second

(*b*) SERIES 2

FIG. 5.—VELOCITY DISTRIBUTION OVER CROSS SECTION OF FLUME
(SEE FIG. 1 FOR CIRCULATION SYSTEMS)

gradient. The fact that this gradient is zero or very small permits the conclusion that near the center of the flume the influence of the walls is negligible.

Fig. 5(*a*), distributions D to F, and Fig. 5(*b*), distributions A to D, give the velocity at various distances from the inlet, and show the gradual stabilization of the velocity. Here again may be noted the improved flow conditions in series 2 (Fig. 5(*b*)). Despite the higher discharge, the equilibrium velocity is established in a short distance and obtains over the remaining length of the flume. The distortion of the velocity at depth 0.5 ft at station 50 (Fig. 5(*b*), distribution D) is due to backwater from the discharge end, which is at station 57. Fig. 5 also shows the effect of the sediment on the flow, discussed subsequently.

TABLE 2.—DATA FOR VELOCITY DISTRIBUTIONS IN FIG. 5

Distribution	Run	Station	Depth y_m	Flow q	Velocity $\bar{u}$	Concentration C_m
(*a*) SERIES 1; CIRCULATION SYSTEM, FIG. 1(*a*)						
A	1	46	0.481	4.80	3.60	0
B	1	46	0.484	4.75	3.54	0.75
C	13	40	0.473	4.87	3.72	3.2
D	1	15	0.485	4.85	3.61	0
E	1	25	0.485	4.85	3.61	0
F	1	46	0.485	4.85	3.61	0
G	1	15	0.480	4.87	3.66	0.75
H	1	26	0.480	4.87	3.66	0.75
I	1	46	0.480	4.87	3.66	0.75
(*b*) SERIES 2; CIRCULATION SYSTEM, FIG. 1(*b*)						
A	14	30	0.554	5.24	3.42	0
B	14	40	....	....	....	....
C	14	45	....	....	....	....
D	14	50	....	....	....	....
E	14	30	0.538	5.24	3.51	1.3
F	14	40	....	....	....	....
G	14	45	....	....	....	....
H	14	50	....	....	....	....
I	20	45	0.46	3.91	2.96	0
J	21	45	0.234	1.29	1.98	0
K	18	45	0.462	2.82	2.20	0
L	17	45	0.269	1.31	1.76	0.05
M	20	45	0.462	4.07	3.18	1.2
N	21	45	0.234	1.42	2.19	1.2
O	18	45	0.462	3.11	2.43	1.2
P	22	45	0.295	2.13	2.60	1.3

Figs. 6 and 7 show dimensionless semilogarithmic graphs of velocity profiles for series 1 and 2, respectively. The relative distance from the bottom $\frac{y}{y_m}$ is plotted on the logarithmic ordinate scale against the dimensionless velocity $\frac{u}{\sqrt{g\,y_m\,S}}$ on the abscissa scale. The quantity $\sqrt{g\,y_m\,S}$ is the so-called friction velocity, commonly written $\sqrt{\frac{\tau_0}{\rho}}$. The abscissa scale is shifted for each curve, the abscissa value 15 being indicated for each curve. The curves in Fig. 6 are arranged in four groups according to the average sediment content of the

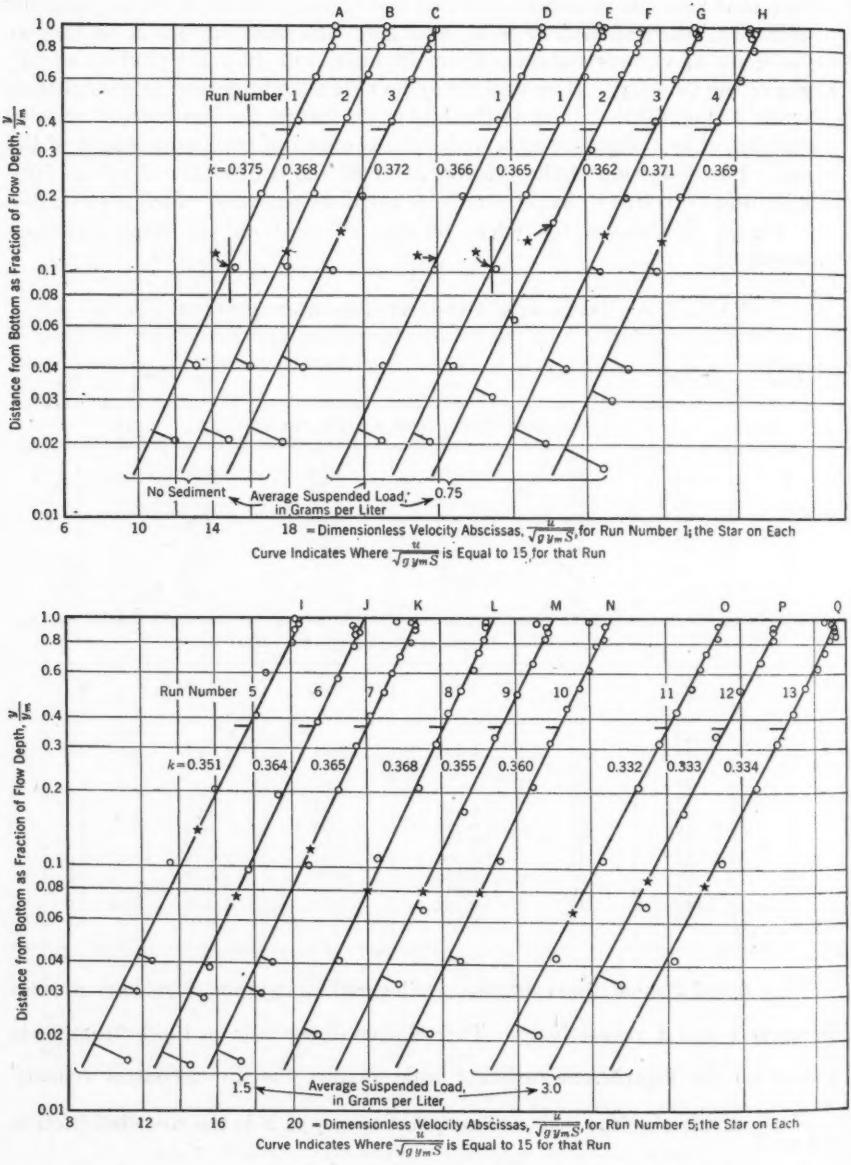
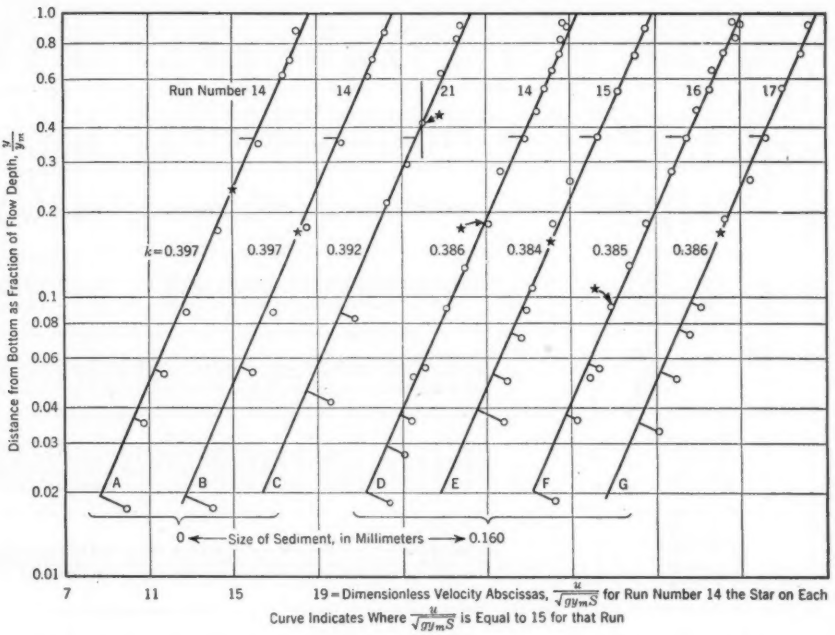

A B C D E F G H
Run Number 1 2 3 1 1 2 3 4
k=0.375 0.368 0.372 0.366 0.365 0.362 0.371 0.369
Distance from Bottom as Fraction of Flow Depth, y/ym
No Sediment
Average Suspended Load, in Grams per Liter
0.75
= Dimensionless Velocity Abscissas, u/√(g ym S), for Run Number 1; the Star on Each Curve Indicates Where u/√(g ym S) is Equal to 15 for that Run
I J K L M N O P Q
Run Number 5 6 7 8 9 10 11 12 13
k=0.351 0.364 0.365 0.368 0.355 0.360 0.332 0.333 0.334
Distance from Bottom as Fraction of Flow Depth, y/ym
1.5
Average Suspended Load, in Grams per Liter
3.0
= Dimensionless Velocity Abscissas, u/√(g ym S), for Run Number 5; the Star on Each Curve Indicates Where u/√(g ym S) is Equal to 15 for that Run

FIG. 6.—SERIES 1, RUNS 1 TO 13; DIMENSIONLESS VELOCITY PROFILES ON CENTER OF FLUME



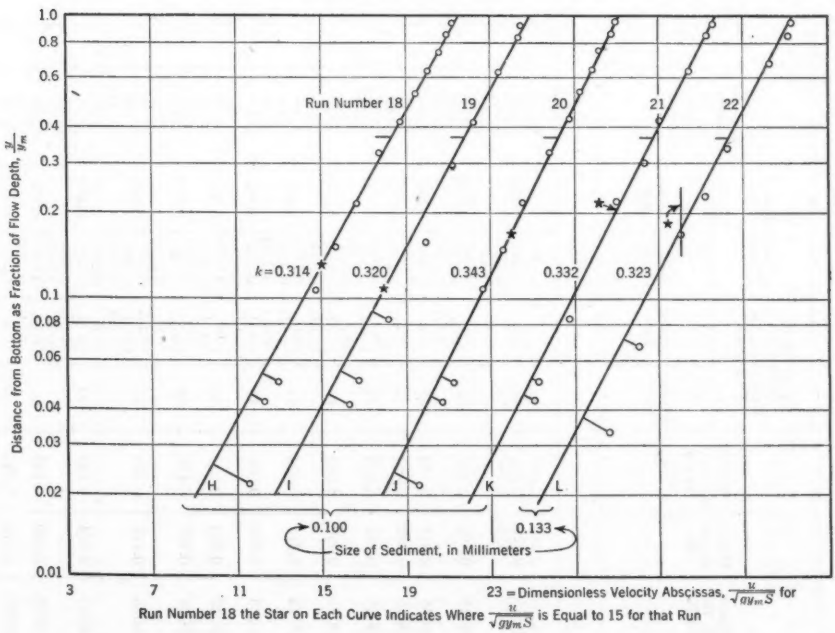


FIG. 7.—SERIES 2, RUNS 14 TO 22; DIMENSIONLESS VELOCITY PROFILES ON CENTER OF FLUME

TABLE 3.—SUMMARY OF EXPERIMENTS

Run No.	Slope S, in ft per ft	Depth y_m, in ft	Friction velocity $\sqrt{g\, y_m\, S}$ (ft per sec)	Measuring Section; Distance, in Ft from: Flume inlet	Measuring Section; Distance, in Ft from: Center line	Universal Constant, k: Individual	Universal Constant, k: Average	Average velocity $\bar{u}_1$, at profile section	Velocity distribution curve[a] (Fig. 6 or 7):	Water temperature, in degrees centigrade	Settling velocity w, in ft per sec	Exponent in Eq. 16: Theoretical z (Eq. 17)	Exponent in Eq. 16: Experimental z_1 (Eq. 18)	Sediment Concentration: Distribution curve[b] (Fig. 9 or 10):	Sediment Concentration: At reference level, C_a	Sediment Concentration: Mean value for flow, C_m
									(a) Series 1 (See Figs. 6 and 9)							
1	0.0025	0.481	0.197	46		0.375		3.62	A	....						
2	0.0025	0.496	0.200	44	0.0	0.368	0.372	3.60	B	....	0.0	0.0	0.0	....	0.0	0.0
3	0.0025	0.481	0.197	44		0.372		3.45	C	....						
					0.0	0.366		3.60	D	23.6	0.0522	0.723	....	....	....	
					0.0	0.365		3.63	E	21.8	0.0485	0.672	0.67	A	2.60	
1	0.0025	0.484	0.197	46	0.5L	0.374	0.370	3.63	....	22.6	0.0492	0.667	0.67	B	2.80	0.72
					0.5R	0.374		3.63	....	21.2	0.0478	0.650	0.67	C	2.62	
2	0.0025	0.314	0.159	46	0.0	0.362	0.362	2.80	F	19.0	0.0450	0.782	0.67	D	2.33	0.35
3	0.0025	0.496	0.200	44	0.0	0.371	0.371	3.52	G	23.3	0.0590	0.795	0.80	E	5.80	0.85
					0.5L	0.371		3.52	....	24.4	0.0602	0.810	0.82	F	5.20	
4	0.0025	0.492	0.199	44	0.0	0.408	0.369	3.63	....	20.8	0.0564	0.767	0.77	H	5.18	0.79
					0.5L	0.369		3.54	H	23.3	0.0590	0.804	0.97	G	5.20	
					0.0	0.351		3.57	I	23.1	0.0590	0.840	0.83	I	8.30	
5	0.0025	0.496	0.200	44	0.5L	0.351	0.351	3.48	....	23.8	0.0596	0.850	0.86	J	6.95	1.59
					1.0L	0.359		3.31	....	24.4	0.0603	0.840	0.86	K	7.70	
6	0.0025	0.518	0.204	44	0.0	0.373	0.369	3.97	J	23.8	0.0596	0.784	0.81	L	9.25	1.78
					0.5L	0.364		3.94	....	26.3	0.0626	0.844	0.83	M	7.35	
					0.0	0.365		3.62	K	25.1	0.0610	0.835	0.85	N	8.40	
7	0.0025	0.498	0.200	44	0.5L	0.371	0.368	3.63	....	26.5	0.0626	0.845	0.91	O	7.00	1.61
					1.0L	0.418		3.43	....	27.4	0.0636	0.761	0.91	P	7.30	
8	0.0025	0.483	0.197	40	0.0	0.363	0.365	3.79	L	26.6	0.0626	0.875	0.87	Q	8.35	1.34
					0.5L	0.368		3.75	....	27.8	0.0636	0.877	0.97	R	8.65	
9	0.0025	0.303	0.156	40	0.0	0.355	0.355	3.03	M	23.7	0.0541	0.977	0.96	S	7.80	1.41
					0.5L	0.355		3.00	....	24.7	0.0541	0.977	0.97	T	7.45	
					0.0	0.360		3.82	N	26.3	0.0626	0.887	....	....	....	....
10	0.0025	0.478	0.196	40	0.5L	0.360	0.360	3.76	....	26.8	0.0626	0.887	....	....	....	....
					1.0L	0.360		3.64	....	27.7	0.0636	0.901	....	....	....	....
					1.0R	0.372		3.50	....	28.5	0.0643	0.882	....	....	....	....
					0.0	0.332		3.97	O	27.0	0.0630	0.973	0.97	V	19.6	
11	0.0025	0.474	0.195	40	0.5L	0.332	0.332	3.85	....	28.2	0.0643	0.993	1.01	W	18.0	3.36
					1.0L	0.332		3.73	....	25.8	0.0620	0.957	1.10	U	22.9	
					0.0	0.333		2.97	P	24.8	0.0568	1.10	1.04	X	12.9	
12	0.0025	0.299	0.155	40	0.5L	0.333	0.333	2.97	....	25.7	0.0578	1.12	1.02	Z	11.9	2.56
					1.0L	0.353		2.85	....	26.2	0.0585	1.07	1.06	Y	11.7	
13	0.0025	0.478	0.196	45	0.0	0.334	0.334	3.81	Q	26.9	0.0580	0.886	0.85	AA	14.0	3.19

TABLE 3.—SUMMARY OF EXPERIMENTS (*Continued*)

Run No.	Slope S, in ft per ft	Depth y_m, in ft	Friction velocity $\sqrt{g\, y_m\, S}$ (ft per sec)	Measuring Section; Distance, in Ft from: Flume inlet	Measuring Section; Distance, in Ft from: Center line	Universal Constant, k: Individual	Universal Constant, k: Average	Average velocity $\bar{u}_1$, at profile section	Velocity distribution curve[a] (Fig. 6 or 7):	Water temperature, in degrees centigrade	Settling velocity w, in ft per sec	Exponent in Eq. 16: Theoretical z (Eq. 17)	Exponent in Eq. 16: Experimental z_1 (Eq. 18)	Sediment Concentration: Distribution curve[b] (Fig. 9 or 10):	Sediment Concentration: At reference level, C_a	Sediment Concentration: Mean value for flow, C_m
								(b) SERIES 2 (SEE FIGS. 7 AND 10)								
14	0.0025	0.554	0.211	40		0.397		3.38	A	....				....		
14	0.0025	0.554	0.211	45	0.0	0.397	0.394	3.39	B	....	0.0	0.0	0.0	....	0.0	0.0
20	0.0025	0.462	0.193	45		0.389		3.21	....	....				....		
21	0.0025	0.234	0.137	45		0.392		1.99	C	....				....		
14	0.0025	0.538	0.208	45	0.0	....	0.386	3.46	D	24.0	0.0590	0.735	0.89	M	9.09	1.33
15	0.0025	0.274	0.148	45	0.0	0.384	0.384	2.54	E	21.0	0.0577	1.02	1.02	N	5.25	0.59
16	0.00125	0.537	0.147	45	0.0	0.385	0.385	2.70	F	21.5	0.0577	1.02	1.05	O	3.02	0.59
						....		....	....	20.0	0.0571	1.42	1.62	S	0.535	
17	0.00125	0.269	0.104	45	0.0	0.386	0.386	1.77	G	20.0	0.0571	1.42	1.08	P	0.036	0.048
						0.386		1.82	....	19.3	0.0567	1.41	1.45	R	0.995	
						....		....	....	19.6	0.0567	1.41	1.47	Q	0.416	
						0.314		2.48	H	19.0	0.0269	0.629	0.53	D	4.79	
18	0.00125	0.462	0.136	45	0.0	0.314	0.317	2.57	....	20.0	0.0275	0.644	0.54	E	4.89	1.18
						0.317		2.49	....	22.0	0.0285	0.660	0.48	C	3.54	
						0.324		2.45	....	21.6	0.0285	0.646	0.54	F	4.25	
				45		0.334		1.78	....	23.2	0.0292	0.897	0.63	G	0.949	
				40		0.334		1.82	....	21.1	0.0282	0.866	0.64	H	0.999	
19	0.00125	0.236	0.0975	40	0.0	0.334	0.330	1.78	....	19.2	0.0272	0.835	0.69	I	1.20	0.17
				45		0.320		1.82	....	20.3	0.0275	0.881	0.68	J	1.44	
				45		0.320		1.82	I	20.5	0.0278	0.891	0.68	K	1.29	
20	0.0025	0.461	0.193	45	0.0	....	0.349	3.24	J	22.0	0.0285	0.423	0.34	A	4.27	....
21	0.0025	0.234	0.136	45	0.0	0.332	0.332	2.28	K	19.7	0.0275	0.609	0.42	B	3.42	....
22	0.0025	0.295	0.154	45	0.0	....	0.341	2.60	L	17.8	0.0407	0.775	0.66	L	6.75	....

[a] See the lettered curves in Fig. 6 for series 1 and Fig. 7 for series 2. [b] See the lettered curves in Fig. 9 for series 1 and Fig. 10 for series 2.

flow. Those in Fig. 7 are grouped according to size of the sediment being transported.

In all runs the velocity distribution fits the logarithmic law very well, although near the bottom there is a systematic deviation from the law, which is not understood. A departure is expected near the bottom since the viscous forces are no longer negligible and the conditions for which the logarithmic law was derived no longer obtain. The viscosity tends to suppress the turbulence and to reduce the exchange coefficient ϵ_m in Eq. 1*b*, and, in order to develop the shear, the velocity gradient must increase. This would cause the velocity to decrease more rapidly than is indicated by the logarithmic law, and the result is opposite to the results shown in Figs. 6 and 7. In the present experiments, the origin of the depth coordinate was assumed to be at the top of the sand grains forming the bottom roughness. An inspection of the velocity profile curves will show that the fit could be improved by assuming the origin below this level although the justification for doing so is not clear. The deviation from the law was first attributed to errors in measurement due to interference between the pitot static tube (diameter, $\frac{3}{16}$ in.) and the bottom. Tubes as small as $\frac{1}{16}$ in. in diameter gave substantially the same readings as the larger instruments and showed that the measurements were reliable. An apparent deviation from the law is also to be noted at the free surface. However, because of surface waves and the disturbance of the pitot tube itself, it is felt that these measurements are not too reliable and that conclusions based on them must be drawn with reservation.

The fact that the velocity profiles fit the logarithmic law so well means that the average velocity will always occur at the relative depth $\frac{y}{y_m} = 0.368$.[29] This point is marked on the curves of Figs. 6 and 7 by a short dash. The average velocities determined by this method agree within 1% or 2% with those determined by graphical integration. The velocities $\bar{U}_1$ in Table 3 were determined by this method.

The velocity curves of Figs. 6 and 7 are of the form,

$$\frac{u}{\sqrt{g\, y_m\, S}} = \frac{u_{\max}}{\sqrt{g\, y_m\, S}} + \frac{2.3}{k} \log \frac{y}{y_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

in which k is a universal constant. The quantity $2.3/k$ is the slope of the curve on the semilogarithmic graph, and should be constant if k does not vary. A close inspection of the curves will show that the slopes do vary, as the size and concentration of sediment are varied. The values of k for each curve are shown in Figs. 6 and 7, and in Table 3. In general, k decreases as the concentration is increased or as the size of sediment in suspension is decreased.

Fig. 8 shows the velocity profiles for series 1 and 2, respectively, plotted according to the von Kármán velocity defect law (Eq. 14). In Fig. 8(*a*), the data of Fig. 6 have been combined according to the sediment content of the flow and show clearly the variation of k. In Fig. 8(*b*), the data of Fig. 7

[29] "Velocity Distribution in Open Channels," by Vito A. Vanoni, *Civil Engineering*, June, 1941, pp. 356–357.

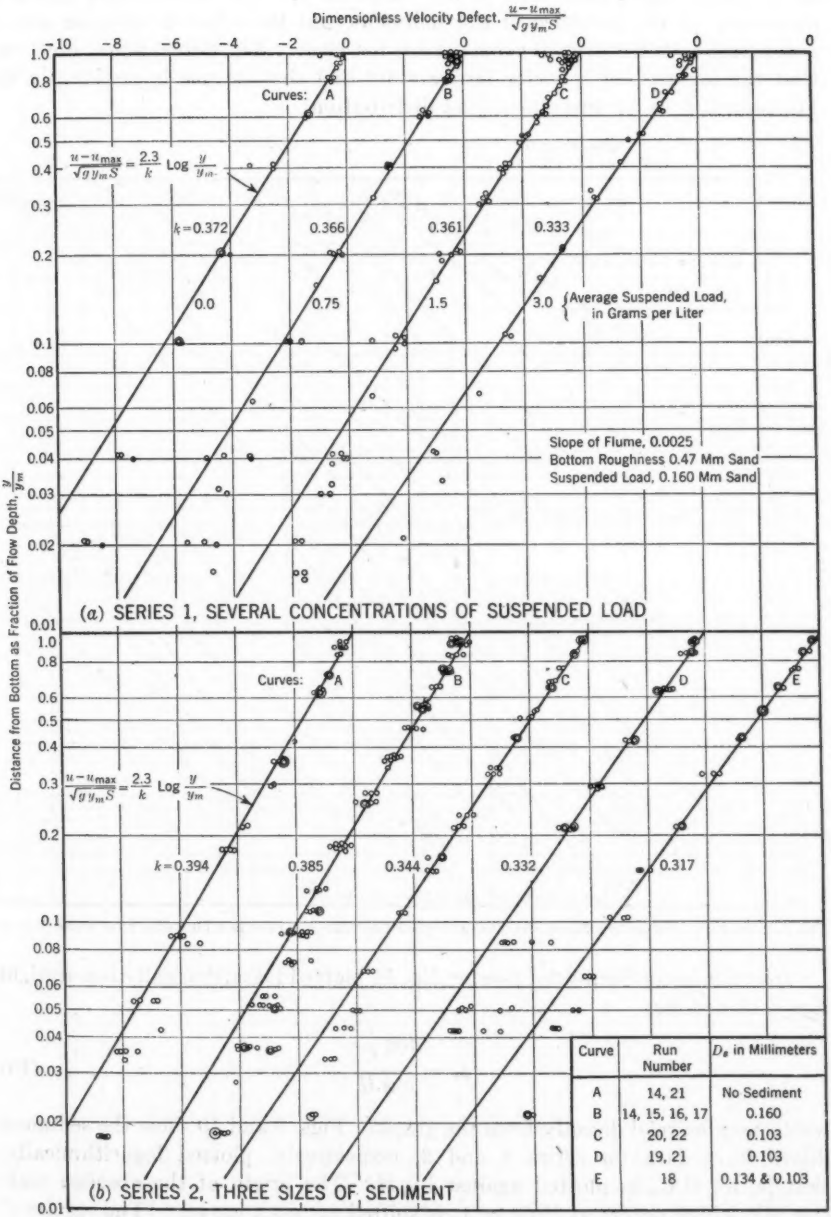

Dimensionless Velocity Defect. $\frac{u-u_{max}}{\sqrt{g\,y_m S}}$
Distance from Bottom as Fraction of Flow Depth, $\frac{y}{y_m}$
Curves: A B C D
$\frac{u-u_{max}}{\sqrt{g\,y_m S}} = \frac{2.3}{k}$ Log $\frac{y}{y_m}$
k=0.372 0.366 0.361 0.333
0.0 0.75 1.5 3.0 Average Suspended Load, in Grams per Liter
Slope of Flume, 0.0025
Bottom Roughness 0.47 Mm Sand
Suspended Load, 0.160 Mm Sand
(a) SERIES 1, SEVERAL CONCENTRATIONS OF SUSPENDED LOAD
Curves: A B C D E
k=0.394 0.385 0.344 0.332 0.317
(b) SERIES 2, THREE SIZES OF SEDIMENT
Curve | Run Number | D_s in Millimeters
A | 14, 21 | No Sediment
B | 14, 15, 16, 17 | 0.160
C | 20, 22 | 0.103
D | 19, 21 | 0.103
E | 18 | 0.134 & 0.103

FIG. 8.—VELOCITY PROFILES

have been grouped according to the values of k. The curves show that k is decreased by the presence of the sediment, but the relation between size of sediment and the universal constant k is not clear. Fig. 8 does show, however, that the logarithmic velocity law is valid and that its use is justified in developing Eq. 16 for suspended load distribution.

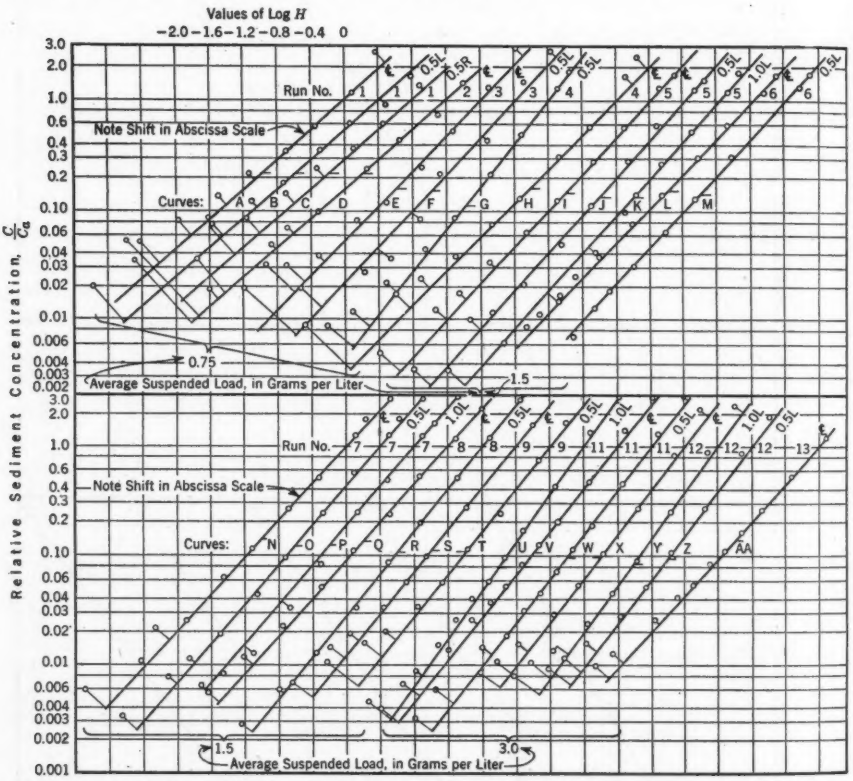


FIG. 9.—SERIES 1; MEASURED SEDIMENT DISTRIBUTION PLOTTED AGAINST THE DISTRIBUTION FUNCTION, H

Distribution of Suspended Load.—Eq. 16 plotted logarithmically is a straight line with a slope,

$$z_1 = \frac{\log \frac{C}{C_a}}{\log H} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

which may be read directly from the graph. Figs. 9 and 10 show the sediment distribution data for series 1 and 2, respectively, plotted logarithmically; that is, $\log C/C_a$ is plotted against $\log H$. The origin of the abscissa scale ($\log H$), which occurs at $C/C_a = 1$, is shifted for each curve. (The results of experiment series 2 were first reported by the writer in 1941.[30])

[30] "Some Experiments on the Transportation of Suspended Load," by Vito A. Vanoni, *Transactions*, Am. Geophysical Union, Pt. III, 1941, pp. 608–621.

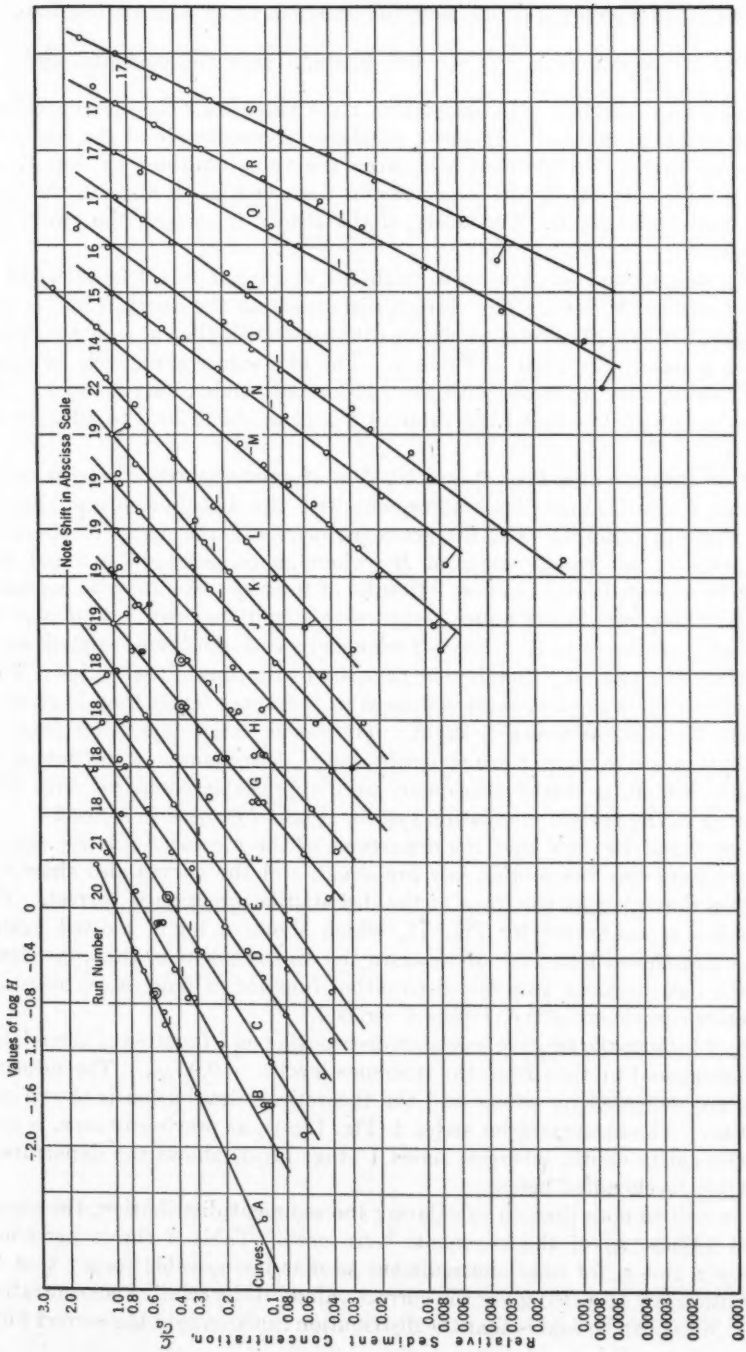


FIG. 10.—SERIES 2; MEASURED SEDIMENT DISTRIBUTION PLOTTED AGAINST THE DISTRIBUTION FUNCTION, *H*

The value of z may also be calculated from Eq. 17, in which w is the settling velocity of the sediment, k is the von Kármán universal constant, and $\sqrt{\frac{\tau_0}{\rho}}$ is the friction velocity. (In calculating z the value $\bar{w}$ of the settling velocity of the center of mass of a group of particles representative of the suspended load was used.) To distinguish between the value obtained by Eq. 17 and that obtained by fitting the experimental data in Figs. 9 and 10, the latter value is denoted by z_1. Obviously, if the theory is correct, the two values will agree.

The intersection of each curve with the line $\log H = 1.0$ is indicated by a short horizontal dash. The vertical distance from the origin ($C/C_a = 1$) to the dash, measured in cycles on the logarithmic scale is the value of z_1. Values of both z and z_1 are listed in Table 3. The two values agree well for runs 1 to 17, which were all made with the coarsest suspended load (sand 1). For all the runs with finer sand the value of z is from about 20% to 40% greater than z_1.

From inspection of Figs. 9 and 10, it is at once apparent that the results of series 2 are in much closer agreement with the distribution law (Eq. 16) than those of series 1. The disagreement occurs in the upper levels of the flow—that is, for small values of H—where more sediment is found than would be expected from the law. Results of the two series may be compared only for those runs made under comparable conditions with sediment of the same size. Runs 14 to 17 of series 2 were made with the same sediment as the runs of series 1 and may therefore be compared with runs of that series. These runs (14 to 17) also show more sediment in the upper levels than is given by Eq. 16, but the difference is slight. The rather large departures from the distribution law in series 1 are no doubt due to disturbances in the flow at the entrance, which, as stated previously, were appreciable compared with those occurring in the revised circulation system (Fig. 1(*b*)), used in series 2.

Figs. 9 and 10 show that the departures of the measured relative concentrations from the distribution law are small, and the curves also show conclusively that at least the form of the distribution function is correct. This conclusion is confirmed by Fig. 11, which shows $z_1 \log H$ plotted against $\log C/C_a$ for series 1 and 2. Here again the disagreement in the upper levels between measurement and the distribution function is noticeable, as is also the better agreement of the results of series 2.

Fig. 12 shows the relative sediment distribution as a function of the relative depth measured upward from the reference level $a = 0.05\, y_m$. The measurements are indicated by circles and the theoretical distribution is shown as a solid line. The agreement for series 2 (Fig. 12(*b*)), as previously seen, is good over the entire depth, whereas, series 1 (Fig. 12(*a*)) shows the departure in the higher levels noted before.

It is well to note that, in calculating the sediment distribution, the experimental values, z_1, of the exponents were used. Table 3 shows agreement between z and z_1 for only one sediment so it is not possible to say that the distribution law (Eq. 16) gives the correct values of the relative concentration, C/C_a. What can be said is that the distribution function is of the correct form.

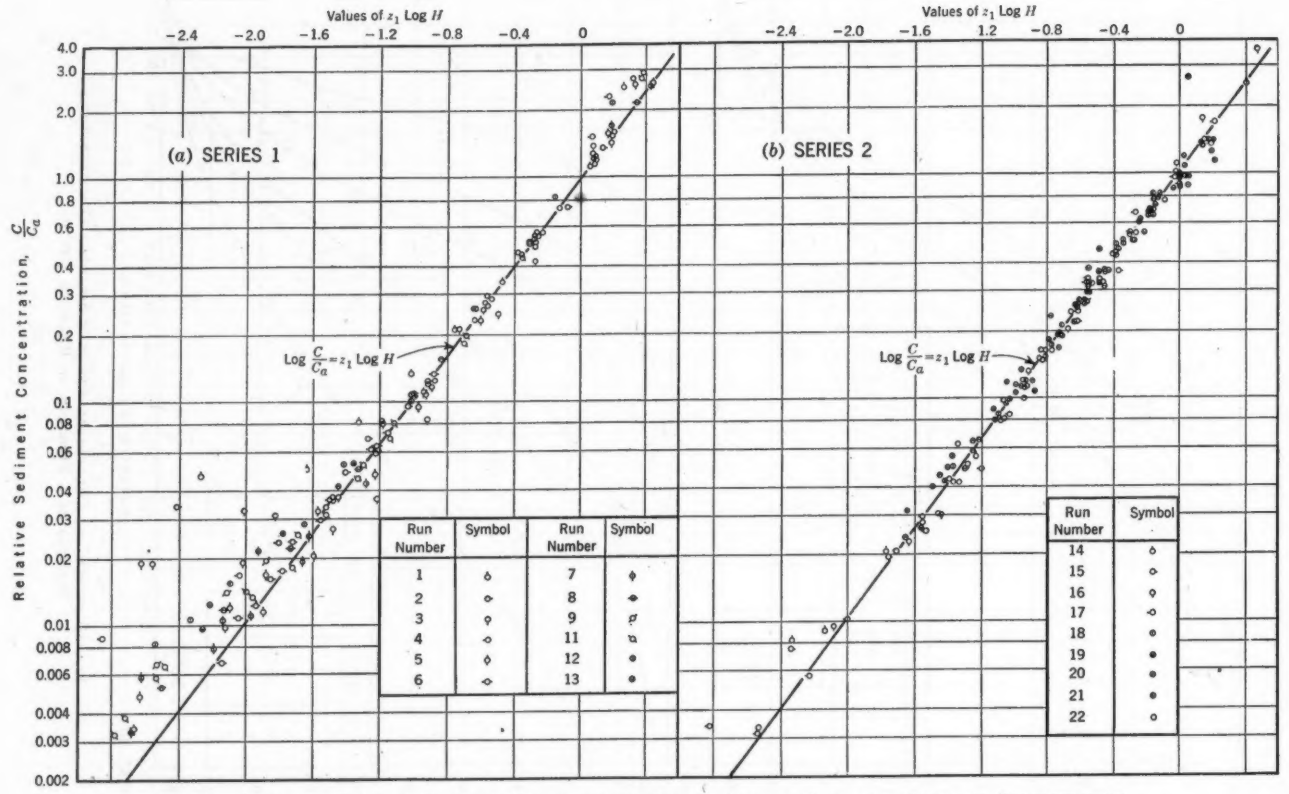

Values of z_1 Log H
−2.4 −2.0 −1.6 −1.2 −0.8 −0.4 0
(a) SERIES 1
(b) SERIES 2
Relative Sediment Concentration, $\frac{C}{C_a}$
4.0 3.0 2.0 1.0 0.8 0.6 0.4 0.3 0.2 0.1 0.08 0.06 0.04 0.03 0.02 0.01 0.008 0.006 0.004 0.003 0.002
Log $\frac{C}{C_a}=z_1$ Log H
Run Number | Symbol | Run Number | Symbol
1 2 3 4 5 6 | 7 8 9 11 12 13
Run Number | Symbol
14 15 16 17 18 19 20 21 22

FIG. 11.—MEASURED RELATIVE SEDIMENT CONCENTRATION PLOTTED AGAINST CALCULATED SEDIMENT CONCENTRATION

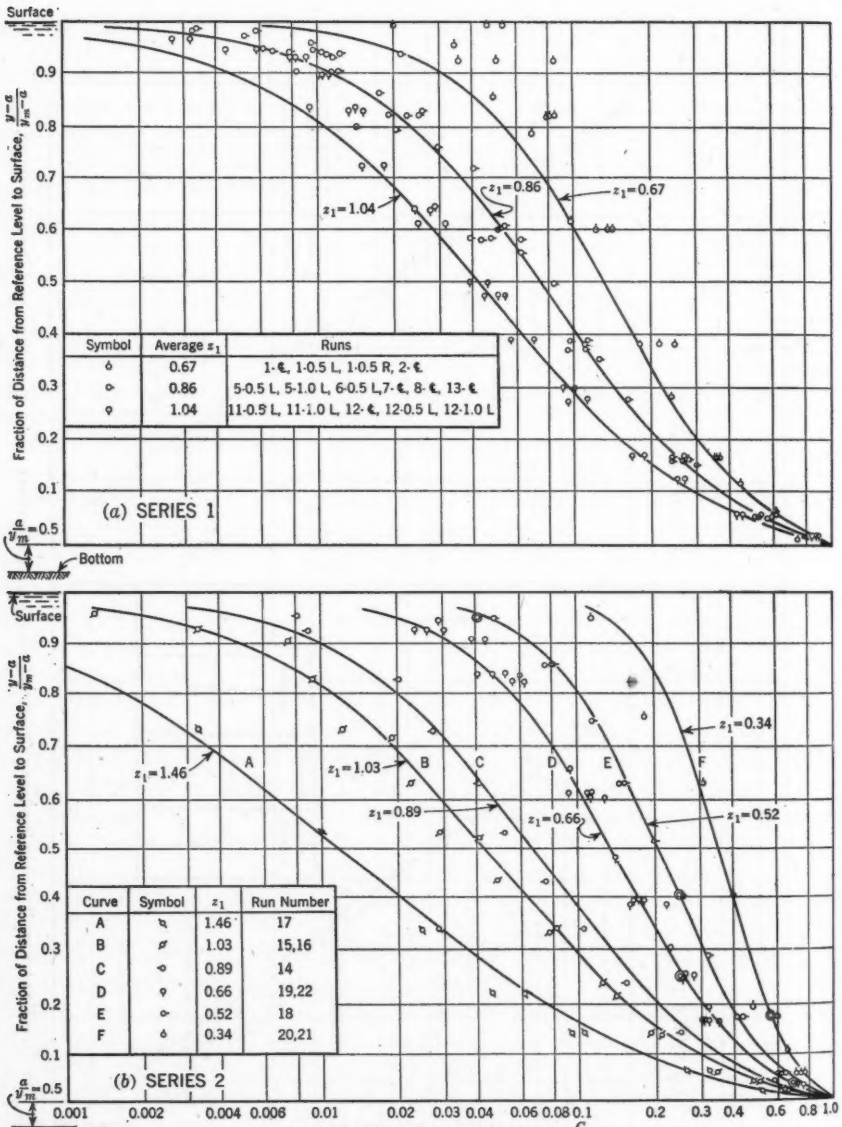


FIG. 12.—MEASURED AND CALCULATED RELATIVE SEDIMENT CONCENTRATIONS PLOTTED AS A FUNCTION OF RELATIVE DEPTH

Turbulent Transfer Coefficients.—The turbulent transfer coefficient for momentum is given by Eq. 10. Introducing Eqs. 12*b* and 15*a* into Eq. 10 gives

$$\epsilon_m = \beta\, v'\, l = k \sqrt{\frac{\tau_0}{\rho}} \left(1 - \frac{y}{y_m} \right) y \qquad (20a)$$

Eq. 20*a* assumes logarithmic velocity distribution and linear distribution of shear and gives the coefficient ϵ_m for momentum transfer. In deriving Eq.16, the assumption was made that $\epsilon_m = \epsilon_s$. The measured sediment distribution now makes it possible to calculate the coefficient ϵ_s for the sediment by applying Eq. 6, which gives

$$\epsilon_s = \frac{w\, C}{\frac{dC}{dy}} \qquad (20b)$$

The transfer coefficients may be reduced to a dimensionless form by dividing by the depth times the friction velocity as follows:

$$\frac{\epsilon_m}{y_m \sqrt{y_m\, g\, S}} = k \left(1 - \frac{y}{y_m} \right) \frac{y}{y_m} \qquad (21a)$$

and

$$\frac{\epsilon_s}{y_m \sqrt{g\, y_m\, S}} = \frac{w}{\sqrt{g\, y_m\, S}} \times \frac{C}{y_m \frac{dC}{dy}} \qquad (21b)$$

Fig. 13 shows both the coefficient for momentum (Eq. 21*a*) and for sediment (Eq. 21*b*) for four flow conditions. Figs. 13(*a*) and 13(*b*) are for series 1 with average suspended loads of 1.34 and 3.36 g per liter (Table 3), respectively, and in general are typical of this series. Figs. 13(*c*) and 13(*d*) are typical of results obtained in series 2, the former being for the coarsest sediment (sand 1).

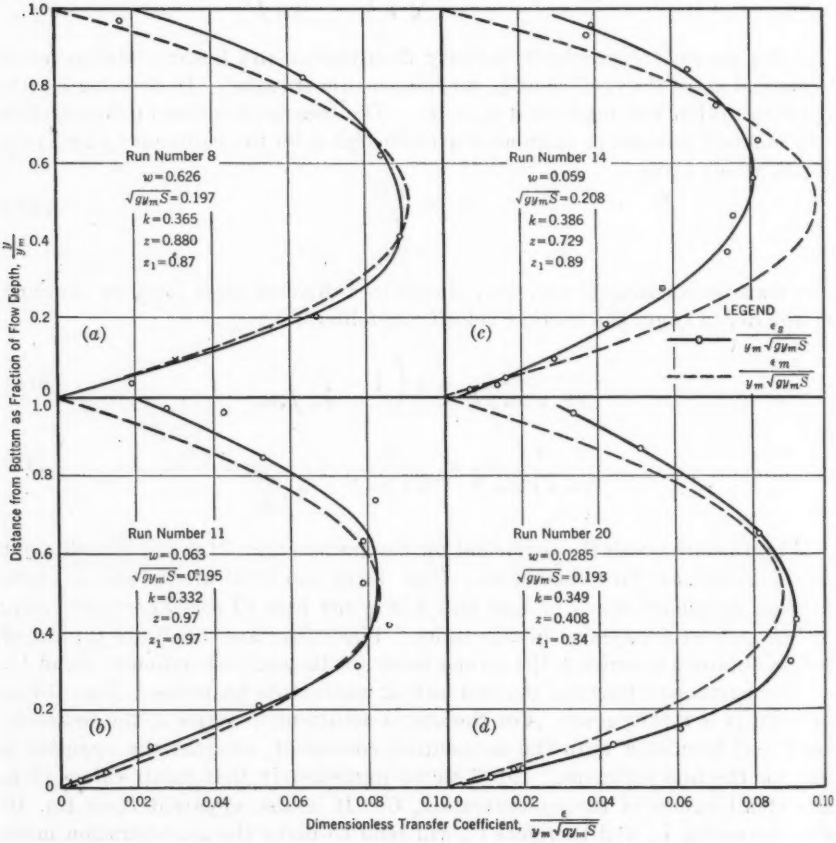
The agreement between the two mixing coefficients for series 1 (Figs. 13(*a*) and 13(*b*)) is rather good. For the coarse sediment in series 2, the sediment coefficient is smaller than the momentum coefficient, whereas the opposite is true for the fine sediment. Eq. 7 shows immediately that small values of ϵ_s give small values of the concentration, C. It is also apparent from Eq. 16 that increasing ϵ_s, and therefore C, will tend to make the concentration more uniform, that is, it will decrease z. These effects may be noted in Figs. 13(*c*) and 13(*d*). In Fig. 13(*c*), ϵ_s is less than ϵ_m and the measured exponent z_1 is greater than z; whereas in Fig. 13(*d*) the opposite is true—z_1 is less than z.

Discussion of Results

Effect of Suspended Load on Velocity Distribution.—The von Kármán universal constant k (Eqs. 14 and 18) depends only on the mechanism of turbulent mixing, and a change in k means that the mixing mechanism has in some way been altered. From Eqs. 1*b* and 15*a*,

$$\epsilon_m = k\, y \frac{\tau}{\rho \sqrt{\frac{\tau_0}{\rho}}} \qquad (22)$$

or the momentum transfer coefficient is directly proportional to k. A decrease in k decreases ϵ_m, and if the shear (depth and slope) is kept constant the velocity gradient will have to increase to compensate for the reduced ϵ_m.



(e) FLOW CHARACTERISTICS

Fig.	Series	Run	w	$\sqrt{g\, y_m\, S}$	k	z	z_1
a	1	8	0.0626	0.197	0.365	0.880	0.87
b	1	11	0.063	0.195	0.332	0.97	0.97
c	2	14	0.059	0.208	0.386	0.729	0.89
d	2	20	0.0285	0.193	0.349	0.408	0.34

FIG. 13.—COEFFICIENTS FOR MOMENTUM TRANSFER AND SEDIMENT TRANSFER

This will result in an increase in the velocity of the flow. Thus it is seen that k is also a factor in establishing the velocity distribution.

The values of k obtained for various sediments and flow conditions are shown in Table 2. The effect upon k of increasing the sediment load is shown

by series 1, runs 1 to 13. As the load, C_m, was increased, the k-values decreased. The results of series 2 show that the coarse sediment (0.160 mm), although reducing k somewhat, has less effect than the finer sediment. In general, k also appears to vary systematically with the friction velocity $\sqrt{g\, y_m\, S}$. However, there appears to be no simple relationship between the several factors. From these results one would expect k to be a function of w, the settling velocity of the sediment, the average sediment load, and the friction velocity $\sqrt{g\, y_m\, S}$.

Attention is called to the lack of agreement between the values of k obtained in series 1 and 2 for similar flows. For instance, the average k for clear water for series 1 was 0.372 whereas for series 2 it was 0.394. No straightforward explanation of these results can be offered. However, the difference is most likely due to the difference in the inlet conditions for the two series of experiments. The k-value obtained for series 2, which was run with very good flow conditions, checks very well with that obtained for pipes, and it is reasonable to assume that the results of this series are more reliable than those of series 1.

The Effect of Suspended Load on the Resistance to Flow.—As stated, a reduction in k results in an increased velocity, which will also give a reduction in the resistance coefficient of the channel. The Chézy formula for flow in conduits is,

$$\bar{u} = c\sqrt{R\, S} \qquad (23)$$

in which $\bar{u}$ is the mean velocity in the cross section, c is the Chézy coefficient, R is the hydraulic mean depth (area of cross section divided by wetted perimeter), and S is the slope or pressure gradient. For open channels, c is given by the Manning formula,

$$c = \frac{1.486}{n} R^{1/6} \qquad (24a)$$

in which n is a roughness coefficient. The Chézy coefficient can also be expressed in terms of f, the resistance coefficient,

$$c = \sqrt{\frac{8\, g}{f}} \qquad (24b)$$

Eqs. 23 and 24 can be solved for n and f with the following result:

$$n = \frac{1.486}{\bar{u}} R^{2/3} S^{1/2} \qquad (25a)$$

and

$$f = \frac{8\, g}{\bar{u}^2} R\, S \qquad (25b)$$

Several values of n and f for flows with and without sediment are shown in Table 4.

The effect of the suspended load in reducing the resistance is seen clearly. The reduction in both n and f appears to be about the same for the two sediments but is greater for the low flows than for the high flows. The first three

lines of Table 4 indicate that the resistance decreases as the sediment load increases.

The increase in velocity or decrease in channel resistance, as suspended load is added to the flow, can be explained by the effect of the sediment in reducing

TABLE 4.—RESISTANCE AND ROUGHNESS COEFFICIENTS OF FLUME WITH AND WITHOUT SUSPENDED LOAD IN FLOW

Run No.	Flow depth, y_m (ft)	Average velocity,[a] $\bar{u}$	Manning's n	Ratio, n: clear flow / silt flow	Resistance coefficient, f	Ratio, f: clear flow / silt flow	Mean diameter, $\bar{D}_s$ (mm)	Average concentration,[b] C_m
....	0.496	3.49	0.0108	....	0.0192	....	....	0
8, 10	0.484	3.59	0.0104	1.04	0.0179	1.07	0.160	1.6
11, 13	0.475	3.64	0.0102	1.06	0.0172	1.12	0.160	3.3
14	0.554	3.42	0.0117	1.04	0.0217	1.08	0.160	0
14	0.538	3.52	0.0112		0.0201			1.33
18	0.462	2.20	0.0118	1.10	0.0231	1.22	0.100	0
18	0.462	2.43	0.0107		0.0189			1.18
20	0.461	3.05	0.0121	1.04	0.0240	1.09	0.100	0
20	0.462	3.18	0.0116		0.0221			1.2
21	0.234	1.99	0.0128	1.10	0.0326	1.22	0.100	0
21	0.234	2.19	0.0116		0.0268			1.2

[a] Feet per second. [b] Grams per liter.

the turbulence. To keep the sediment from settling, work must be done on it. The energy for this work can be provided only by the turbulence which is damped and reduced in intensity when it gives up energy. This means that the momentum transfer coefficient is also decreased thus allowing the velocity and velocity gradient to increase according to Eq. 1*b*. (The late Henry M. Eakin, formerly head of the Sedimentation Section, Division of Research, Soil Conservation Service, had anticipated the effect of the sediment on the flow and advanced his idea on the subject to the writer in 1936 during a discussion of the suspended-load problem.)

A. B. Buckley[31] observed, from measurements on the Nile River in Egypt, that suspended load decreased the roughness coefficient of the channel. The reduction in resistance that he obtained for sediment loads of from 1.0 to 2.0 g per liter was in general greater than that obtained in the experiments described in this paper. Mr. Buckley attributed the reduction in resistance to the damping effect upon the turbulence of a heavily loaded layer of water near the bed of the stream. He expected the greatest reduction when a stream was depositing, for then the layer near the bottom would be most dense.

Distribution of Suspended Load.—All the measurements of suspended-load distribution followed very closely the form of the sediment distribution law, Eq. 16, but agreement between the calculated and measured value of the exponent z occurred only for the coarsest suspended load ($\bar{D}_s = 0.160$ mm). For the finer sediments the value z_1 of the exponent that fits the data, or the measured value, was about 20% less than the value calculated from the theory. (Measurements of suspended load in a natural stream made recently by Alvin G. Anderson,[32] Assoc. M. Am. Soc. C. E., show that z_1 was about half as

[31] "The Influence of Silt on the Velocity of Flowing Water in Open Channels," by A. B. Buckley, *Minutes of Proceedings*, Inst. C. E., Vol. 216, 1922–1923, pp. 183–211.

[32] "Distribution of Suspended Sediment in a Natural Stream," by Alvin G. Anderson, *Transactions*, Am. Geophysical Union, Pt. II, 1942, pp. 678–683.

large as z.) As noted previously, the smaller the value of z the more uniformly the sediment is distributed over the depth.

The mixing coefficient ϵ is given by Eq. 3 and has the same form for both momentum transfer and sediment transfer but not necessarily the same value.[5] For a fine sediment the mixing coefficient for momentum was smaller than the corresponding coefficient for sediment transfer. It can readily be seen that increasing ϵ_s will tend to produce a more uniform sediment distribution, which results in z_1 being smaller than z.

The fact that the coefficients of sediment transfer and of momentum transfer are not the same may be explained by two differences in the transfer mechanisms for sediment and momentum. The first difference occurs because, for momentum transfer, a correlation is required between the horizontal and vertical turbulent velocity fluctuations and uncorrelated or random fluctuations do not contribute to the momentum transfer. For sediment transfer this kind of correlation is not necessary and the random fluctuations also transfer sediment and tend to make ϵ_s larger than ϵ_m. The second difference exists because of the "slip" between the fluid and the sediment. The theory assumes that the instantaneous vertical velocity of the sediment relative to the fluid is always equal to its settling velocity w in still fluid. This is true only for steady conditions and during acceleration the relative velocity must be greater than w and the sediment is carried a shorter distance than assumed. This will tend to make ϵ_s less than ϵ_m.

The experiments show that in flows carrying the coarsest sediment the two transfer coefficients are about equal and that, as finer sediment is introduced, ϵ_s becomes increasingly larger relative to ϵ_m. Apparently for the fine sediment the effect of the random turbulence in increasing ϵ_s is greater than the effect of the "slip" between the particles and fluid in decreasing ϵ_s. For the coarse sediment the two effects are about balanced and if still coarser sediment were used a resultant effect of opposite sign is expected and ϵ_s should be less than ϵ_m. The effect of the variation of the transfer coefficients relative to one another is reflected in the relative values of z and z_1 which may be noted in Table 1, For flows carrying the fine sediment where ϵ_s is greater than ϵ_m, z_1 is less than z. and when the two transfer coefficients are equal, z and z_1 are also equal.

The tendency for the sediment content near the surface of the flow to exceed that indicated by the distribution formula, Eq. 16, can be explained by the action of the random turbulence. Although the mixing coefficient for momentum goes to zero at the surface, the random velocity fluctuations carry sediment right up to the very surface, thus giving the mixing coefficient for sediment a finite value at the surface. This can be noted in Fig. 13.

Attention is called to the wide disagreement between the values of z and z_1 in the first two sediment profiles of run 17 (Table 3). Considerable difficulty was experienced in this run because the sediment was not distributed in a regular manner over the cross section. Since the flow was very low and the sand coarse, considerable material was moving along the bottom. Any irregularity in the distribution was therefore easily noticed. The first measurement (Table 3 and curve S, Fig. 10) was made over a point where considerable sediment was in motion in a narrow band on the bottom. The abnormally

large value of z_1 could be accounted for by the occurrence of a transfer of sediment in the horizontal direction away from the measuring point, brought about by the existence of a concentration gradient across the flow. In the same way it is possible to account for the low value of z_1 obtained in the second measurement of run 17. This measurement was taken over a point where very little sediment was in motion on the bed but a few inches to one side of the section considerable sand was being carried. The transverse transfer of sediment into the measuring section would decrease the vertical concentration gradient and therefore the value of z_1. It is surprising that even under such conditions the distribution should follow the law expressed by Eq. 7.

Secondary Flow Due to Suspended Load.—The effect of the suspended load was first noticed in measuring the velocity distribution over cross sections of the flume. Symmetrical velocity distributions in clear flows became distorted by the addition of very small amounts of sand. This effect may be noted in Fig. 5(*a*), distributions A, B, and C, which show the velocity distributions for clear water and for average sediment concentrations of 0.75 and 3.0 g per liter, respectively. To give an idea how small the concentrations actually were, it might be well to remark that the first concentration ($C_m = 0.75$) was obtained by adding only 12.5 lb of sand to the entire system. The effect of this small concentration on the flow may also be seen by comparing the center column of Fig. 5(*a*) with the right column. The effect is small for this small concentration, but it is noticeable. Incidentally, attention is called to the uniform spacing of the velocity lines in Fig. 5(*a*), distribution A. This shows at once that the velocity distribution is logarithmic since the distances y from the bottom of the flume are spaced geometrically.

The disturbing effect of the sediment is also shown in Fig. 5(*b*). Distributions A to D, Fig. 5(*b*), for clear water, may be compared with distributions E to H, Fig. 5(*b*), which show the velocities under like conditions except that there is sediment in the flow. Distributions I and M, Fig. 5(*b*), show a case in which the sediment apparently has a stabilizing effect. In this one instance, the conditions were apparently just right to establish good flow at this particular station. The fact that the distorted velocity distribution with clear water was made more uniform by the addition of a small amount of sediment serves to illustrate again that the sediment has considerable influence on the flow.

The sediment appeared to be transported in clouds or ribbons parallel to the flow. At low flows when appreciable material was moving along the bottom, these ribbons appeared as bands of sand on the bottom. When proper flow conditions had been attained, these ribbons were symmetrically spaced about the center line and were stable regardless of the flow rate. Attempts to break them up artificially were only partly successful. Fig. 14 is a photograph of these bands of sediment taken after run 15, looking upstream. The photograph was taken after the flow had been stopped and the water slowly drained off. Material in suspension settled to the bottom, thus showing how it was distributed across the flow. Three main streaks occur in the center portion of the flume and two less pronounced streaks occur at the walls. This pattern, which was symmetrical, occurred over the entire length of the

FIG. 15.—FLOW CHARACTERISTICS, RIVER PO IN ITALY

FIG. 14.—SEDIMENT ON FLOOR OF FLUME AS DEPOSITED BY THE FLOW OF RUN NO. 15

flume. Measurement of suspended load at a distance of 0.02 ft from the bottom showed that the sediment was very nearly uniformly distributed over the center half of the flume, that it then suddenly dropped to about 25% of the center concentration as the clear areas near the walls (Fig. 14) were approached, and finally increased to about 50% of the center concentration near the walls. The concentration for the center 6 in. was about 5% less than that over the concentrated streaks at the side. (The flume is 33¼ in. wide by 12 in. deep.)

The cause of these longitudinal bands of sediment is not known, although it is believed that they are the result of secondary circulation and disturbances due to the presence of suspended load. Even the slightest departure from a uniform sediment distribution will cause horizontal density gradients that can set up secondary flows. This distribution shows a concentration gradient across the flow, and one could then expect a circulation with an upward flow at the points of low concentration and a downward flow where the load was high. Since there appear to be four points of low concentration, four pairs of circulations might be expected. No direct observations were made to attempt to check this hypothesis, and it is doubtful that this could be done without considerable work. However, this is a possible explanation of the action of the sediment in causing secondary circulations in the flow. Further investigation will show whether or not this idea is valid.

If circulation could be caused by sediment in a uniform and straight laboratory channel, there is a very good possibility that it will also occur in natural streams where irregularities and disturbances always exist. Therefore, some measurements of suspended load in rivers were examined. Figs. 15(*a*) and 15(*b*) show measurements of velocity and suspended load at a cross section of the River Po in Italy, reported by Giandotti.[33]

Note the high concentration of suspended load at about the one-quarter points of the section. The circulation expected from this distribution is shown in Fig. 15(*c*). The writer is not able to state whether this distribution is typical of rivers. However, these measurements in the River Po are evidence that a circulation may be caused by the sediment. The circulation that would be induced is the same as that advanced as an explanation for the depression of the maximum velocity in natural streams. Although this circulation explains the velocity distribution, no satisfactory explanation was ever found for the circulation itself. The action of sediment on river flow may well prove to be an important factor in determining the velocity distribution.

Sediment-Transporting Capacity.—The idea that sediment in suspension reduces the turbulence furnishes a possible mechanism for establishing the capacity of a stream. Since the turbulence supports the sediment a reduction in the turbulence by adding sediment means that the stream then is less capable of supporting sediment than before a load was added. The capacity load is probably determined by continually decreasing the turbulence as more sediment is taken into suspension. The fact that the addition of sediment to

[33] "Les Déplacements du Lit du Pô," by Mario Giandotti, Le Service Hydrographique Italien, XV Congres International De Navigation, Venice, September, 1931, pp. 297–322.

a flow increases the velocity, therefore, does not mean that the sediment transporting capacity is also increased. On the contrary the capacity is actually decreased. The increase in velocity also means that the depth and, hence, the boundary shear is decreased and that particles in the bed are probably subjected to less entrainment force than when the velocity was lower.

It is seen not only that turbulence is involved in the sediment transportation problem, but that the sediment changes the turbulence itself, the amount of change apparently depending upon the amount and size of the suspended load. In a clear flow the velocity can be expressed in terms of certain constants which characterize the turbulence mechanism. In sediment-laden flows these constants are now functions of the sediment load and sediment characteristics.

Conclusions

1. The distribution of the relative concentration of suspended load has the form of the derived distribution law (Eq. 16) but does not agree quantitatively with it; that is, the value of the exponent z given by the theory does not agree with z_1, the exponent that fits the experimental results. When the suspended sediment is fine, z_1 is less than z. This shows that the relative concentration is more uniformly distributed and has a greater average value than that indicated by the theory. In other words, a flow can support and, therefore, transport fine sediment more effectively than the theory indicates. As the size of the sediment is increased, the agreement between the two exponents improves up to a certain point, beyond which the value of z_1 may be expected to exceed z.

2. The disagreement between the calculated and measured distributions cited in conclusion 1 is attributed to (*a*) the action of the random turbulent fluctuations in suspending sediment, and (*b*) the "slip" between the fluid and the sediment as the sediment is accelerated. The theory is based on the assumption that the coefficient of sediment transfer is equal to the coefficient of momentum transfer, and therefore does not account for the action of the purely random turbulence and of the slip. Such disagreement was anticipated, aud the present results confirm it.

3. For fine material the coefficient of sediment transfer tends to exceed the coefficient of momentum transfer; for coarser sediment the opposite tendency is found. Conclusion 3 may be considered a corollary of conclusion 1, the physical reason for the conclusions being set forth in conclusion 2.

4. Suspended load decreases the coefficient of momentum transfer, the reduction varying with the amount of the load and being largest for fine sediment.

5. Suspended load reduces the value of the von Kármán universal constant of turbulent exchange k, which characterizes the effectiveness of the turbulence in transferring momentum. Reduction of k means that the mixing is less effective and would indicate that the sediment tends to suppress or damp out the turbulence. This conclusion follows directly from conclusion 4.

6. Suspended load reduces the resistance to flow, thus causing sediment-laden water to flow faster than clear water. This follows directly from con

clusions 4 and 5. In the laboratory channel an average suspended load of 1.2 g per liter reduced the friction factor, f, as much as 20%, and the corresponding Manning roughness coefficient was reduced as much as 10%.

7. Conclusions 1 to 6 can all be related to essentially three effects that occur in the flow in the presence of the sediment: (1) The sediment appears to damp out the turbulence in such a way that the momentum transfer is reduced, (2) random turbulence, which is not a factor in the transfer of momentum, contributes to the transfer of sediment, and (3) the "slip" between the fluid and the sediment tends to make the sediment transfer coefficient less than the momentum transfer coefficient.

8. The idea that suspended sediment damps the turbulence suggests that the transporting capacity of a stream is determined through the reduction of the turbulence by the sediment.

9. The increase in velocity, due to the addition of sediment to a clear stream, probably does not give rise to an increased sediment-transporting capacity since it is accompanied by a reduction in turbulence and in the entrainment force at the stream bed.

10. Suspended load tends to cause a flow to become unevenly distributed. Unsymmetrical sediment distribution within the flow may cause secondary circulation.

11. In rivers, density gradients due to suspended load may be an important cause of the secondary circulation, which has the effect of causing the maximum velocity to occur below the water surface.

12. The velocity distribution near the center of the flume used in these experiments was of the logarithmic form and followed the von Kármán universal velocity defect law.

Acknowledgments

The writer wishes to acknowledge the friendly assistance of Professors von Kármán and Knapp and of members of the staff of the Cooperative Laboratory of the Soil Conservation Service and the California Institute of Technology in which the experiments reported herein were performed. Acknowledgment is due also to Roy Peterson, Milton Wood, and Robert Wise for their assistance in collecting and analyzing the data.

Assistance in the preparation of these data was furnished by the personnel of Work Projects Administration, Official Project No. 65–2–07–58, Work Projects No. PS–11496.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

CHARACTERISTIC REDUNDANTS USED FOR ANALYZING STATICALLY INDETERMINATE STRUCTURES

By John B. Wilbur,[1] Assoc. M. Am. Soc. C. E.

Synopsis

Orthogonal functions are commonly used in the solution of vibration problems. To a lesser degree such functions have been applied to some of the more complicated static analyses, as, for example, to plate analysis and to the determination of beam deflections by trigonometric series. In this paper, forces acting on relatively simple structures are reduced to characteristic forces which are orthogonal, and methods of analysis based on these characteristic forces are discussed.

The heart of the multiply indeterminate solution may be interpreted by means of the determination of so-called characteristic forces. After these forces have been found, redundant stresses corresponding to any condition of loading or distortion may be evaluated without the further solution of simultaneous equations, regardless of the degree to which the structure may be statically indeterminate.

It is not claimed that the methods suggested are either more quickly applied or more easily understood than other methods which are commonly used. It is hoped, however, that they will be of interest because they differ so radically from the usual methods. Being so different, it is possible that other applications of the principles involved may lead to methods of solution which constitute an improvement on the methods outlined herein.

Notation

The symbols used in this paper are defined where they first appear, either by text or diagram, and are assembled for convenience of reference in the Appendix.

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by November 1, 1944.

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1. Characteristic Forces

Suppose an elastic structure to be acted upon by a number of concentrated forces $P_a, P_b, \cdots, P_n$, which are applied at points a, b, $\cdots$, n, respectively, as shown in Fig. 1. Let:

Δ_a = total deflection at point a in the direction of P_a due to forces $P_a, P_b, \cdots, P_n$;

Δ_b = total deflection at point b in the direction of P_b due to forces $P_a, P_b, \cdots, P_n$;

. . . . . .

Δ_n = total deflection at point n in the direction of P_n due to forces $P_a, P_b, \cdots, P_n$.

Let values of $P_a, P_b, \cdots, P_n$ be determined in such a manner that

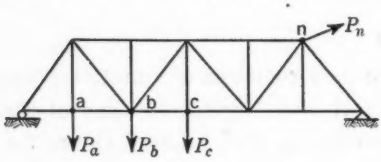

FIG. 1

$$\left.\begin{aligned} \Delta_a &= \kappa P_a \\ \Delta_b &= \kappa P_b \\ &\cdots\cdots \\ \Delta_n &= \kappa P_n \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

in which κ is a proportionality factor having the same value in each of Eqs. 1. A set of forces which satisfy Eqs. 1 will be called "characteristic forces."

2. Determination of Characteristic Forces

If δ_{mn} is the deflection at any point m in the direction of P_m, due to a unit value of P_n, then, by superposition:

$$\left.\begin{aligned} P_a\,\delta_{aa} + P_b\,\delta_{ab} + \cdots + P_n\,\delta_{an} &= \Delta_a \\ P_a\,\delta_{ba} + P_b\,\delta_{bb} + \cdots + P_n\,\delta_{bn} &= \Delta_b \\ &\cdots\cdots\cdots\cdots \\ P_n\,\delta_{na} + P_b\,\delta_{nb} + \cdots + P_n\,\delta_{nn} &= \Delta_n \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (2)$$

Substituting values of $\Delta_a, \Delta_b, \cdots, \Delta_n$ from Eqs. 1, Eqs. 2 may be rewritten as

$$\left.\begin{aligned} P_a\,(\delta_{aa} - \kappa) + P_b\,\delta_{ab} + \cdots + P_n\,\delta_{an} &= 0 \\ P_a\,\delta_{ba} + P_b\,(\delta_{bb} - \kappa) + \cdots + P_n\,\delta_{bn} &= 0 \\ &\cdots\cdots\cdots\cdots \\ P_a\,\delta_{na} + P_b\,\delta_{nb} + \cdots + P_n\,(\delta_{nn} - \kappa) &= 0 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (3)$$

Since the right-hand side of Eqs. 3 is zero, the numerator determinant will be zero. Hence, in order for the forces $P_a, P_b, \cdots, P_n$ to have values other than zero, it is necessary that the denominator determinant formed by the coefficients to $P_a, P_b, \cdots, P_n$ on the left side of these equations be also equal to zero. This condition is represented by

$$\begin{vmatrix} \delta_{aa} - \kappa; & \delta_{ab} & ; \cdots; & \delta_{an} \\ \delta_{ba} & ; \delta_{bb} - \kappa; & \cdots; & \delta_{bn} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{na} & ; \delta_{nb} & ; \cdots; & \delta_{nn} - \kappa \end{vmatrix} = 0 \quad \cdots\cdots\cdots\cdots (4)$$

The expansion of the determinant of Eq. 4 leads to an nth degree algebraic equation in terms of κ. If n forces are being considered, there will be n values of κ which will satisfy this equation. These n values of κ will be denoted by $\kappa_1, \kappa_2, \cdots, \kappa_n$, taken in order of descending magnitude.

Each of these values of κ_m may now be substituted separately into Eqs. 3. The simultaneous solution of any $n - 1$ equations of Eqs. 3 will then permit the expression of any $n - 1$ of the forces $P_{am}, P_{bm}, \cdots, P_{nm}$ in terms of any one of these forces such as P_{am}. The nth equation of Eqs. 3 may be used to check both κ_m and the solution of the $n - 1$ equations. Since there are n values of κ_m, n sets of characteristic forces exist, which are determined by the foregoing procedure. The complete set of characteristic forces corresponding to κ_m will be denoted by P_m.

There will result then, the following n sets of characteristic forces:

$$\left.\begin{array}{l}\text{For } \kappa_1,\\ \qquad P_{a1} = P_{a1};\quad P_{b1} = A_{b1}\,P_{a1};\quad \cdots;\quad P_{n1} = A_{n1}\,P_{a1}\\ \text{for } \kappa_2,\\ \qquad P_{a2} = P_{a2};\quad P_{b2} = A_{b2}\,P_{a2};\quad \cdots;\quad P_{n2} = A_{n2}\,P_{a2}\\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \text{and, for } \kappa_n,\\ \qquad P_{an} = P_{an};\quad P_{bn} = A_{bn}\,P_{an};\quad \cdots;\quad P_{nn} = A_{nn}\,P_{an}\end{array}\right\} \quad \cdots\cdots\cdots (5)$$

In Eqs. 5, A_{mn} is a factor by which P_{an} is multiplied to obtain P_{mn}, values of A_{mn} having been determined by the simultaneous solution of $n - 1$ equations of Eqs. 3.

3. Normalizing of Characteristic Forces

In Section 2 methods were presented whereby each set of characteristic forces P_m could be expressed in terms of any one force of the set, such as P_{am}. The relations of Eqs. 5 do not define P_{am}, however, which may be given any value without altering the characteristic aspect of the loads. As a matter of mathematical convenience in the development to follow, P_{am} is chosen arbitrarily so that:

$$(P_{am})^2 + (P_{bm})^2 + \cdots + (P_{nm})^2 = +1 \quad \cdots\cdots\cdots\cdots (6)$$

To satisfy Eq. 6, substitute values of P_m from Eqs. 5 and solve for P_{am}:

$$P_{am} = \frac{1}{\sqrt{1 + (A_{bm})^2 + \cdots + (A_{nm})^2}} \quad \cdots\cdots\cdots\cdots (7)$$

When values of P_{am} are determined in this manner, a set of characteristic forces is said to be "normalized." In the following treatment, normalized characteristic forces will be denoted by Q. Thus Q_{mn} represents the normalized characteristic force at m corresponding to $\kappa = \kappa_n$. Similarly, Q_m represents the complete set of normalized characteristic forces for points a, b, $\cdots$, n, corresponding to $\kappa = \kappa_m$. Using this new notation, Eqs. 7, 5, and 6, respectively,

should be rewritten as

$$Q_{am} = \frac{1}{\sqrt{1 + (A_{bm})^2 + \cdots + (A_{nm})^2}} \quad \ldots\ldots\ldots\ldots\ldots (8)$$

For κ_1,

$$Q_{a1} = Q_{a1};\quad Q_{b1} = A_{b1}\,Q_{a1};\quad \cdots;\quad Q_{n1} = A_{n1}\,Q_{a1}$$

for κ_2,

$$Q_{a2} = Q_{a2};\quad Q_{b2} = A_{b2}\,Q_{a2};\quad \cdots;\quad Q_{n2} = A_{n2}\,Q_{a2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \ldots\ldots\ldots (9)$$

for κ_n,

$$Q_{an} = Q_{an};\quad Q_{bn} = A_{bn}\,Q_{an};\quad \cdots;\quad Q_{nn} = A_{nn}\,Q_{an}$$

and (see Eq. 6):

$$(Q_{am})^2 + (Q_{bm})^2 + \cdots + (Q_{nm})^2 = +1 \quad \ldots\ldots\ldots\ldots (10)$$

4. Orthogonality of Normalized Characteristic Forces

Suppose that a structure acted upon by one set of normalized characteristic forces Q_m is subjected to a virtual distortion corresponding to the deformation that occurs when a second set of normalized characteristic forces Q_n is applied to the structure. The external virtual work done by the forces Q_m may be expressed by

$$W_{mn} = Q_{am}\,\Delta_{an} + Q_{bm}\,\Delta_{bn} + \cdots + Q_{nm}\,\Delta_{nn} = \kappa_n\,(Q_{am}\,Q_{an} + Q_{bm}\,Q_{bn} + \cdots + Q_{nm}\,Q_{nn}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Now suppose that the same structure, when acted upon by the forces Q_n, is subjected to a virtual distortion corresponding to the deformation that occurs when the forces Q_m are applied to the structure. The external virtual work done by forces Q_n may be expressed by

$$W_{nm} = Q_{an}\,\Delta_{am} + Q_{bn}\,\Delta_{bm} + \cdots + Q_{nn}\,\Delta_{nm} = \kappa_m\,(Q_{an}\,Q_{am} + Q_{bn}\,Q_{bm} + \cdots + Q_{nn}\,Q_{nm}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Betti's law[2] states that the virtual work done by a group of external forces R_m during a distortion due to a group of external forces R_n is equal to the virtual work done by the forces R_n during a distortion due to the forces R_m. Hence, W_{mn} and W_{nm} can be equated from Eqs. 11 and 12, respectively, leading to

$$\kappa_n\,(Q_{am}\,Q_{an} + Q_{bm}\,Q_{bn} + \cdots + Q_{nm}\,Q_{nn}) = \kappa_m\,(Q_{an}\,Q_{am} + Q_{bn}\,Q_{bm} + \cdots + Q_{nn}\,Q_{nm}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Since the parenthetical expressions on the two sides of Eq. 13 are identical, and, since $\kappa_n \neq \kappa_m$, it follows that the equation can be satisfied only when

$$Q_{am}\,Q_{an} + Q_{bm}\,Q_{bn} + \cdots + Q_{nm}\,Q_{nn} = 0 \quad \ldots\ldots\ldots\ldots (14)$$

Thus, for any two sets of normalized characteristic forces, the sum of the cross products of forces applied at the same point vanishes. Thus the two sets of

[2] "Theory of Statically Indeterminate Structures," by W. M. Fife and J. B. Wilbur, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 39.

normalized characteristic forces are orthogonal; and orthogonality does not depend on normalizing.

5. First Virtual Work Theorem

Because of the orthogonality of normalized characteristic forces, it follows from Eq. 11 that if $n \neq m$,

$$W_{mn} = \kappa_n(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Thus the external virtual work done by any one set of normalized characteristic forces Q_m, during a distortion caused by any other set of normalized characteristic forces Q_n, is equal to zero. This conclusion will be called the first virtual work theorem for normalized characteristic forces. Its validity does not depend on normalizing.

6. Second Virtual Work Theorem

Assuming that $n = m$, it follows from Eq. 11 that

$$W_{mm} = \kappa_m [(Q_{am})^2 + (Q_{bm})^2 + \cdots + (Q_{nm})^2] = \kappa_m \quad \ldots\ldots\ldots (16)$$

since the forces Q_m are normalized. Thus the virtual work done by a given set of normalized characteristic forces Q_m, during a distortion caused by the same set of normalized characteristic forces Q_m, is equal to κ_m. This conclusion will be called "the second virtual work theorem for normalized characteristic forces." Its validity does depend on normalizing.

7. Expression of Actual Forces in Terms of Normalized Characteristic Forces

Referring to Fig. 1, let $R_a, R_b, \cdots, R_n$ be any set of actual forces applied at points a, b, $\cdots$, n, respectively, along the lines of action of the characteristic forces $P_a, P_b, \cdots, P_n$, respectively. These actual forces may be expressed in terms of normalized characteristic forces by superposition as follows:

$$\left.\begin{aligned} R_a &= B_1 Q_{a1} + B_2 Q_{a2} + \cdots + B_n Q_{an} \\ R_b &= B_1 Q_{b1} + B_2 Q_{b2} + \cdots + B_n Q_{bn} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R_n &= B_1 Q_{n1} + B_2 Q_{n2} + \cdots + B_n Q_{nn} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (17)$$

in which $B_1, B_2, \cdots, B_n$ may be called the participation factors for the first, second, $\cdots$, and nth sets of normalized characteristic forces, respectively, with respect to forces $R_a, R_b, \cdots, R_n$. If $R_a, R_b, \cdots, R_n$ are known, the participation factors may be evaluated as follows:

With the forces $R_a, R_b, \cdots, R_n$ acting on the structure, suppose the structure to undergo a virtual distortion due to the further application of the mth set of normalized characteristic forces Q_m. The virtual movements of points a, b, $\cdots$, n will be given respectively by $\Delta_{am} = \kappa_m Q_{am}$; $\Delta_{bm} = \kappa_m Q_{bm}$; $\cdots$; $\Delta_{nm} = \kappa_m Q_{nm}$. The virtual work done by forces $R_a, R_b, \cdots, R_n$ is given by

$$W_{Rm} = R_a \Delta_{am} + R_b \Delta_{bm} + \cdots + R_n \Delta_{nm} = \kappa_m (R_a Q_{am} + R_b Q_{bm} + \cdots + R_n Q_{nm}) \quad \ldots\ldots\ldots\ldots (18)$$

Substituting the relations of Eqs. 17 into Eq. 18 and simplifying:

$$\begin{aligned} W_{Rm} = + \kappa_m \{ & B_1 (Q_{a1} Q_{am} + Q_{b1} Q_{bm} + \cdots + Q_{n1} Q_{nm}) \\ & + B_2 (Q_{a2} Q_{am} + Q_{b2} Q_{bm} + \cdots + Q_{n2} Q_{nm}) \\ & + \cdots \cdots \cdots \cdots \cdots \cdots \\ & + B_m [(Q_{am})^2 + (Q_{bm})^2 + \cdots + (Q_{nm})^2] \\ & + B_n (Q_{an} Q_{am} + Q_{bn} Q_{bm} + Q_{nn} Q_{nm}) \} \end{aligned} \quad \ldots\ldots\ldots (19a)$$

From the relations of Eqs. 14 and 10 it follows that

$$W_{Rm} = \kappa_m B_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19b)$$

Equating Eqs. 18 and 19*b* for W_{Rm} and solving for B_m,

$$B_m = R_a Q_{am} + R_b Q_{bm} + \cdots + R_n Q_{nm} \quad \ldots\ldots\ldots\ldots (20)$$

With values of B_m determined by Eq. 20, the superposition of Eqs. 17 is completely defined.

8. Reciprocal Relations

Letting B_m as defined by Eq. 20 be substituted into the mth equation of Eqs. 17, and simplifying,

$$\begin{aligned} R_m = & R_a (Q_{a1} Q_{m1} + Q_{a2} Q_{m2} + \cdots + Q_{an} Q_{mn}) \\ & + R_b (Q_{b1} Q_{m1} + Q_{b2} Q_{m2} + \cdots + Q_{bn} Q_{mn}) \\ & + \cdots \cdots \cdots \cdots \cdots \cdots \\ & + R_m [(Q_{m1})^2 + (Q_{m2})^2 + \cdots + (Q_{mn})^2] \\ & + R_n (Q_{n1} Q_{m1} + Q_{n2} Q_{m2} + \cdots + Q_{nn} Q_{mn}) \end{aligned} \quad \ldots\ldots\ldots (21)$$

Eq. 21 must hold irrespective of the values assigned to $R_a, R_b, \cdots, R_n$. Assigning a value to R_m but making all other values of R equal to zero:

$$(Q_{m1})^2 + (Q_{m2})^2 + \cdots + (Q_{mn})^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots (22)$$

If R_n be assigned a value, but all other values of R be taken as zero:

$$Q_{n1} Q_{m1} + Q_{n2} Q_{m2} + \cdots + Q_{nn} Q_{mn} = 0 \quad \ldots\ldots\ldots\ldots (23)$$

In a sense, Eqs. 22 and 23 are reciprocal expressions to Eqs. 10 and 14.

9. Use of Normalized Characteristic Forces in Computing Deflections

Referring to Section 7, let it be required to compute δ_m, the total deflection of point m in the direction of R_m, due to known actual forces $R_a, R_b, \cdots, R_n$. By Eqs. 17 these actual forces may be replaced by the characteristic forces $B_1 Q_1, B_2 Q_2, \cdots, B_n Q_n$. The deflection of point m due to the normalized characteristic forces Q_1 is given by $\Delta_{m1} = \kappa_1 Q_{m1}$. Hence, by superposition,

$$\delta_m = B_1 \kappa_1 Q_{m1} + B_2 \kappa_2 Q_{m2} + \cdots + B_n \kappa_n Q_{mn} \quad \ldots\ldots\ldots (24)$$

10. Choice of Characteristic Redundants

Let it be required to find the redundant stresses $X_a, X_b, \cdots, X_n$ in an indeterminate structure such as that shown in Fig. 2(*a*), due to any condition of

external loading or imposed strains such as those caused by temperature changes or pier settlements. These redundant stresses may be considered as external loads acting on a statically determinate primary structure, as shown in Fig. 2(*b*), the values of $X_a, X_b, \cdots, X_n$ being so determined that $\delta_a, \delta_b, \cdots, \delta_n$, which represent the relative movements of points a, b, ⋯, n, respectively, due to all causes, are in each instance equal to zero.

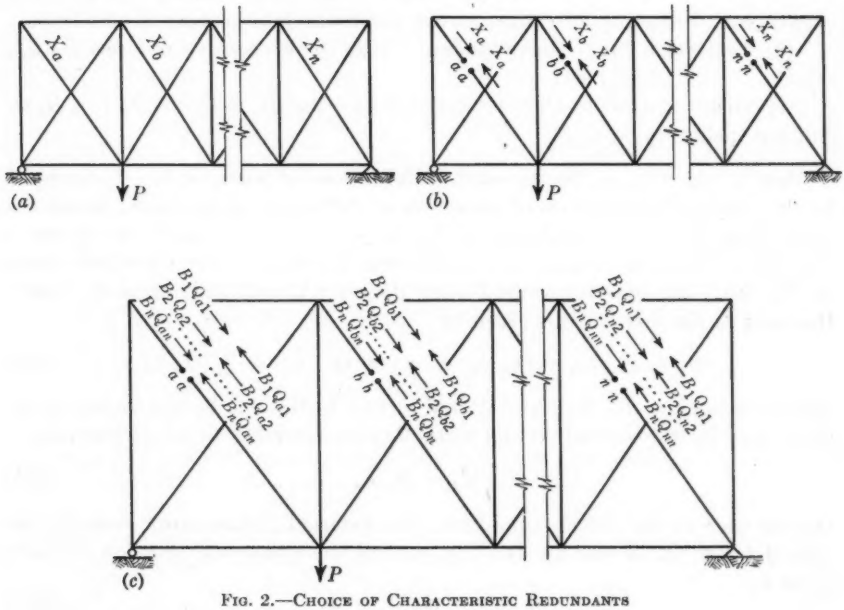

FIG. 2.—CHOICE OF CHARACTERISTIC REDUNDANTS

The redundants $X_a, X_b, \cdots, X_n$, however, may be expressed in terms of normalized characteristic forces, in accordance with Eqs. 17 as follows:

$$\left.\begin{aligned} X_a &= B_1 Q_{a1} + B_2 Q_{a2} + \cdots + B_n Q_{an} \\ X_b &= B_1 Q_{b1} + B_2 Q_{b2} + \cdots + B_n Q_{bn} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ X_n &= B_1 Q_{n1} + B_2 Q_{n2} + \cdots + B_n Q_{nn} \end{aligned}\right\} \cdots\cdots\cdots\cdots (25)$$

Thus one may replace the redundant forces shown in Fig. 2(*b*) with the characteristic forces shown in Fig. 2(*c*), wherein, if the normalized characteristic forces $Q_1, Q_2, \cdots, Q_n$ have previously been determined, the participation factors $B_1, B_2, \cdots, B_n$ are the unknowns that must be determined in order that $X_a, X_b, \cdots, X_n$ may be computed from Eqs. 25.

11. EVALUATION OF PARTICIPATION FACTORS

Since $X_a, X_b, \cdots, X_n$ are not immediately known, values of $B_1, B_2, \cdots, B_n$ cannot be computed by Eq. 20. One may proceed as follows, however:

To evaluate B_m, apply the mth set of normalized characteristic forces Q_m, consisting of $Q_{am}, Q_{bm}, \cdots, Q_{nm}$ at points a, b, ⋯, n, respectively, as virtual

forces that act on the primary structure during its distortion due to all causes combined. Note that Q_{am} consists of a pair of equal and opposite forces applied at points a, etc.

The distortion of the primary structure may be considered as due to three classes of causes:

(*a*) Distortion due to external forces, temperature changes, pier settlements, etc., including all causes except the redundant stresses themselves;

(*b*) Distortion due to the redundant $B_m\ Q_m$ (which consists of forces $B_m\ Q_{am}$, $B_m\ Q_{bm}$, $\cdots$, $B_m\ Q_{nm}$); and

(*c*) Distortion due to all other redundants—that is, $B_1\ Q_1$, $B_2\ Q_2$, $\cdots$, $B_n\ Q_n$, but not including $B_m\ Q_m$.

Let δ_{ao}, δ_{bo}, $\cdots$, δ_{no} be the relative deflections of points a, b, $\cdots$, n, respectively, due to the first class of causes, these deflections being taken as positive when they act in the directions of X_a, X_b, $\cdots$, X_n, respectively, as shown in Fig. 2(*b*). Then, assuming the virtual forces Q_m to act in the directions shown in Fig. 2(*c*), the external virtual work done by the virtual forces Q_m during this part of the distortion is given by

$$W_1 = Q_{am}\,\delta_{ao} + Q_{bm}\,\delta_{bo} + \cdots + Q_{nm}\,\delta_{no} = \sum(Q_m\,\delta_{mo}) \quad \ldots\ldots (26a)$$

Due to redundant $B_m\ Q_m$, the external virtual work done by the virtual forces Q_m is seen by the second virtual work theorem (Section 6) to be given by

$$W_2 = B_m\,\kappa_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26b)$$

Due to each of the other redundants, the external virtual work done by the virtual forces Q_m is seen by the first virtual work theorem (Section 5) to be given by

$$W_3 = B_m(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26c)$$

The total external virtual work done by the virtual forces Q_m is given by $W_1 + W_2 + W_3$. This total must equal zero since points a, b, $\cdots$, n undergo no relative movements when all causes are considered. Hence $\sum(Q_m\,\delta_{mo} + B_m\,\kappa_m) = 0$; and

$$B_m = -\frac{\sum(Q_m\,\delta_{mo})}{\kappa_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

For structures in direct stress, if F_{nm} is the stress in any bar of the primary structure due to a unit value of Q_{nm}, and if ΔL is the change of length of any bar of the primary structure due to external loads or temperature changes,

$$\begin{aligned}\sum(Q_m\,\delta_{mo} &= Q_{am}\,\delta_{ao} + Q_{bm}\,\delta_{bo} + \cdots + Q_{nm}\,\delta_{no})\\ &= Q_{am}\sum(\Delta L\,F_{am} + Q_{bm})\sum(\Delta L\,F_{bm} + \cdots + Q_{nm})\sum(\Delta L\,F_{nm})\\ &= \sum[\Delta L\,(Q_{am}\,F_{am} + Q_{bm}\,F_{bm} + \cdots + Q_{nm}\,F_{nm})] \quad \ldots\ldots (28a)\end{aligned}$$

but $Q_{am}\,F_{am} + Q_{bm}\,F_{bm} + \cdots + Q_{nm}\,F_{nm}$ equals the stress in any bar of the primary structure due to all the virtual forces Q_m acting simultaneously, and may be denoted by F_m, whence

$$\sum(Q_m\,\delta_{mo}) = \sum(\Delta L\,F_m) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28b)$$

Considering only the action of external forces, ΔL is given by

$$\Delta L = \frac{F_o L}{A E} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

in which F_o is the stress in any bar of the primary structure due to external forces only. Therefore,

$$\sum (Q_m \delta_{mo}) = \sum \left(\frac{F_o F_m L}{A E} \right) \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

and Eq. 27 becomes

$$B_m = -\frac{1}{\kappa_m} \sum \left(\frac{F_o F_m L}{A E} \right) \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Thus, $B_1, B_2, \cdots, B_n$ may be evaluated by Eq. 27 or 31. Then the redundant stresses $X_a, X_b, \cdots, X_n$ may be determined by Eqs. 25.

It may be noted that, although the determination of characteristic loads involves the solution of Eqs. 3, the evaluation of redundant stresses corresponding to any condition of loading or distortion involves no further solution of simultaneous equations, regardless of the degree to which the structure is statically indeterminate.

12. Illustrative Solution

The use of characteristic redundants in analyzing a statically indeterminate structure is illustrated by an example evaluating the stresses X_a, X_b, and X_c in

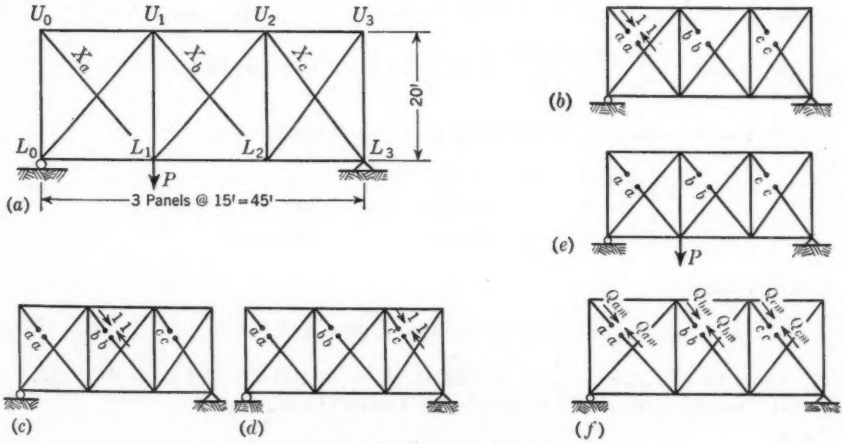

Fig. 3.—Illustrative Solution

the structure shown in Fig. 3(*a*). For convenience, assume that for each member, $\frac{L}{A E} = 25$.

Determination of Normalized Characteristic Loads.—Four steps are required to normalize the characteristic loads: (*a*) Evaluation of δ_{mn}; (*b*) evaluation of κ_m; (*c*) determination of characteristic loads; and (*d*) normalizing.

(*a*) Evaluation of δ_{mn} (By Method of Virtual Work).—Let F_{am}, F_{bm}, and F_{cm} be the stresses in any bar due to the condition of loading shown in Figs. 3(*b*), 3(*c*), and 3(*d*), respectively. The computations for δ_{mn} are arranged as shown in Table 1. For example, $\delta_{aa} = \sum\left[\frac{(F_{am})^2 L}{A E}\right] = \sum 25\,(F_{am})^2 = +100.$

TABLE 1.—EVALUATION OF δ_{mn}

Bar	$5\,F_{am}$	$5\,F_{bm}$	$5\,F_{cm}$	$25\,(F_{am})^2$	$25\,(F_{bm})^2$	$25\,(F_{cm})^2$	$25\,F_{am}\,F_{bm}$	$25\,F_{am}\,F_{cm}$	$25\,F_{bm}\,F_{cm}$
L_0L_1	−3	0	0	+ 9	0	0	0	0	0
L_1L_2	0	−3	0	0	+ 9	0	0	0	0
L_2L_3	0	0	−3	0	0	+ 9	0	0	0
U_0U_1	−3	0	0	+ 9	0	0	0	0	0
U_1U_2	0	−3	0	0	+ 9	0	0	0	0
U_2U_3	0	0	−3	0	0	+ 9	0	0	0
U_0L_0	−4	0	0	+16	0	0	0	0	0
U_1L_1	−4	−4	0	+16	+16	0	+16	0	0
U_2L_2	0	−4	−4	0	+16	+16	0	0	+16
U_3L_3	0	0	−4	0	0	+16	0	0	0
U_0L_1	+5	0	0	+25	0	0	0	0	0
L_0U_1	+5	0	0	+25	0	0	0	0	0
U_1L_2	0	+5	0	0	+25	0	0	0	0
L_1U_2	0	+5	0	0	+25	0	0	0	0
U_2L_3	0	0	+5	0	0	+25	0	0	0
L_2U_3	0	0	+5	0	0	+25	0	0	0
Σ =	....	....	....	+100	+100	+100	+16	0	+16

Similarly $\delta_{bb} = +100$; and $\delta_{cc} = +100$. Also, $\delta_{ab} = \delta_{ba} = \sum\left(\frac{F_{am}\,F_{bm}\,L}{A\,E}\right)$ $= \sum 25\,F_{am}\,F_{bm} = +16$; $\delta_{ac} = \delta_{ca} = \sum\left(\frac{F_{am}\,F_{cm}\,L}{A\,E}\right) = \sum 25\,F_{am}\,F_{cm} = 0$; and $\delta_{bc} = \delta_{cb} = \sum\left(\frac{F_{bm}\,F_{cm}\,L}{A\,E}\right) = \sum 25\,F_{bm}\,F_{cm} = +16.$

(*b*) Evaluation of κ_m (Using Eq. 4).—Eq. 4 becomes—

$$\begin{vmatrix} +100-\kappa; & +16 ; & 0 ; \\ +16 ; & +100-\kappa; & +16 ; \\ 0 ; & +16 ; & +100-\kappa; \end{vmatrix} = 0$$

which reduces to

$$(100-\kappa)\,(+9{,}488 - 200\,\kappa + \kappa^2) = 0 \quad \ldots\ldots\ldots\ldots (32)$$

The roots to Eq. 32 are: $\kappa_1 = +122.63$; $\kappa_2 = +100.00$; and $\kappa_3 = +77.37$.

(*c*) Determination of Characteristic Loads (Using Eqs. 3).—

For κ_1,

$$\left.\begin{aligned} &(+100.00 - 122.63)\,P_{a1} + 16\,P_{b1} = 0; \\ &P_{b1} = +\frac{22.63}{16}\,P_{a1} = 1.414\,P_{a1} = \sqrt{2}\,P_{a1}; \qquad (A_{b1} = +\sqrt{2}) \\ &+16\,P_{b1} + (+100.00 - 122.63)\,P_{c1} = 0; \\ &P_{c1} = +\frac{16\,P_{b1}}{22.63} = \frac{P_{b1}}{\sqrt{2}} = \frac{\sqrt{2}\,P_{a1}}{\sqrt{2}} = P_{a1}; \qquad (A_{c1} = +1) \end{aligned}\right\} \ldots (33a)$$

For κ_2,

$$\left.\begin{aligned} &(100.00 - 100.00)\, P_{a2} + 16\, P_{b2} = 0; \\ &P_{b2} = \frac{0}{16} P_{a2} = 0; \qquad (A_{b2} = 0) \\ &+ 16\, P_{a2} + (100.00 - 100.00)\, P_{b2} + 16\, P_{c2} = 0; \\ &P_{c2} = -\frac{16}{16} P_{a2} = -P_{a2}; \qquad (A_{c2} = -1) \end{aligned}\right\} \ldots (33b)$$

For κ_3,

$$\left.\begin{aligned} &(+100.00 - 77.37)\, P_{a3} + 16\, P_{b3} = 0; \\ &P_{b3} = -\frac{22.63}{16} P_{a3} = -\sqrt{2}\, P_{a3}; \qquad (A_{b3} = -\sqrt{2}) \\ &+16\, P_{b3} + (+100.00 - 77.37)\, P_{c3} = 0; \\ &P_{c3} = -\frac{16}{22.63} P_{b3} = -\frac{P_{b3}}{\sqrt{2}} = -\frac{-\sqrt{2}\, P_{a3}}{\sqrt{2}} = +P_{a3}; \qquad (A_{c3} = +1) \end{aligned}\right\} \ldots (33c)$$

Hence the characteristic loads are defined by:

$$\left.\begin{aligned} &P_{a1} = +P_{a1}; \quad P_{b1} = +\sqrt{2}\, P_{a1}; \quad P_{c1} = +P_{a1} \\ &P_{a2} = +P_{a2}; \quad P_{b2} = 0 \quad ; \quad P_{c2} = -P_{a2} \\ &P_{a3} = +P_{a3}; \quad P_{b3} = -\sqrt{2}\, P_{a3}; \quad P_{c3} = +P_{a3} \end{aligned}\right\} \ldots\ldots\ldots (34)$$

(*d*) Normalizing of Characteristic Loads.—Substituting in Eq. 8, $Q_{a1} = \frac{1}{\sqrt{1+2+1}} = +\frac{1}{2}$. Similarly, $Q_{a2} = +\frac{1}{\sqrt{2}}$; and $Q_{a3} = +\frac{1}{2}$. Hence, by Eq. 9, the normalized characteristic loads are given by:

$$\left.\begin{aligned} &Q_{a1} = +\tfrac{1}{2} \quad ; \quad Q_{b1} = +\frac{\sqrt{2}}{2}; \quad Q_{c1} = +\tfrac{1}{2} \\ &Q_{a2} = +\frac{\sqrt{2}}{2}; \quad Q_{b2} = 0 \quad ; \quad Q_{c2} = -\frac{\sqrt{2}}{2} \\ &Q_{a3} = +\tfrac{1}{2} \quad ; \quad Q_{b3} = -\frac{\sqrt{2}}{2}; \quad Q_{c3} = +\tfrac{1}{2} \end{aligned}\right\} \ldots\ldots\ldots (35)$$

Evaluation of Redundants.—The first step is to determine the value of B_m. Let F_o and F_m be the stress in any bar due to the condition of loading shown in Figs. 3(*e*) and 3(*f*), respectively. Note that the F_m-stresses may be obtained from the F_{am}, F_{bm}, and F_{cm} stresses given in Table 1 by superposition. The arrangement of the computation is shown in Table 2, which supplies the remaining values to be substituted in Eq. 31, thus:

$$B_m = -\frac{1}{\kappa_m} \sum \left(\frac{F_o F_m L}{A E} \right) = -\frac{1}{\kappa_m} \sum \left(\frac{120\, F_o F_m}{120} \right) \times 25$$

$$= -\frac{5}{24\, \kappa_m} \sum (120\, F_o F_m) \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

Hence, $B_1 = \dfrac{-5(-34P + 18\sqrt{2}P)}{24(+122.63)} = +0.0145\ P$. Similarly, $B_2 = +0.4890\ P$ and $B_3 = +0.1601\ P$. The redundants are then determined by substitution in Eqs. 25, thus:

$$X_a = +0.0145\,P\,(+\tfrac{1}{2}) + 0.4890\,P\left(+\frac{\sqrt{2}}{2}\right) + 0.1601\,P\,(+\tfrac{1}{2}) = +0.4330\,P.$$

Similarly, $X_b = -0.1030\,P$ and $X_c = -0.2586\,P$.

TABLE 2.—EVALUATION OF B_m

Bar	$12\,F_o$	$10\,F_1$	$10\,F_2$	$10\,F_3$	$120\,F_o\,F_1$	$120\,F_o\,F_2$	$120\,F_o\,F_3$
L_0L_1	$+6P$	-3	$-3\sqrt{2}$	-3	$-18P$	$-18\sqrt{2}P$	$-18P$
L_1L_2	$+3P$	$-3\sqrt{2}$	0	$+3\sqrt{2}$	$-9\sqrt{2}P$	0	$+9\sqrt{2}P$
L_2L_3	0	-3	$+3\sqrt{2}$	-3	0	0	0
U_0U_1	0	-3	$-3\sqrt{2}$	-3	0	0	0
U_1U_2	$-6P$	$-3\sqrt{2}$	0	$+3\sqrt{2}$	$+18\sqrt{2}P$	0	$-18\sqrt{2}P$
U_2U_3	$-3P$	-3	$+3\sqrt{2}$	-3	$+9P$	$-9\sqrt{2}P$	$+9P$
L_0U_0	0	-4	$-4\sqrt{2}$	-4	0	0	0
L_1U_1	$+8P$	$-4-4\sqrt{2}$	$-4\sqrt{2}$	$-4+4\sqrt{2}$	$-32P$	$-32\sqrt{2}P$	$-32P+32\sqrt{2}P$
L_2U_2	$-4P$	$-4\sqrt{2}-4$	$+4\sqrt{2}$	$+4\sqrt{2}-4$	$+16P$	$+16\sqrt{2}P$	$-16\sqrt{2}P+16P-16\sqrt{2}P$
L_3U_3	$-4P$	-4	$+4\sqrt{2}$	-4	$+16P$	$-16\sqrt{2}P$	$+16P$
U_0L_1	0	$+5$	$+5\sqrt{2}$	$+5$	0	0	0
L_0U_1	$-10P$	$+5$	$+5\sqrt{2}$	$+5$	$-50P$	$-50\sqrt{2}P$	$-50P$
U_1L_2	0	$+5\sqrt{2}$	0	$-5\sqrt{2}$	0	0	0
L_1U_2	$+5P$	$+5\sqrt{2}$	0	$-5\sqrt{2}$	$+25\sqrt{2}P$	0	$-25\sqrt{2}P$
U_2L_3	0	$+5$	$-5\sqrt{2}$	$+5$	0	0	0
L_2U_3	$+5P$	$+5$	$-5\sqrt{2}$	$+5$	$+25P$	$-25\sqrt{2}P$	$+25P$
$\Sigma =$	....	....	....	....	$-34P+18\sqrt{2}P$	$-166\sqrt{2}P$	$-34P-18\sqrt{2}P$

It will be noted that the additional computation required to investigate a second condition of loading is not very great, since modifying the F_o-stresses will change only a part of the computation for B_m and consequently only a part of the final step.

13. ALTERNATE METHOD OF ANALYZING STATICALLY INDETERMINATE STRUCTURES BY MEANS OF CHARACTERISTIC FORCES

If the number of independent components of displacement for a structure is less than the degree to which the structure is statically indeterminate, it may be desirable to modify the details of the analysis.

Characteristic forces which correspond to the independent components of distortion may be determined and normalized. For the actual external loading condition to be investigated, participation factors for the normalized characteristic forces may be evaluated. Deflections of the points of application of these forces may then be computed, following which, the evaluation of stresses in the structure is straightforward.

Suppose, for example, it is required that the end moments in the frame of Fig. 4(*a*) be determined, due to the load P acting as shown. Corresponding

to the three independent components of distortion θ_A, θ_B, and d, there will be considered as a set of characteristic forces the external moments X_a, X_b, and the external force X_c, respectively, acting as shown in Fig. 4(*b*).

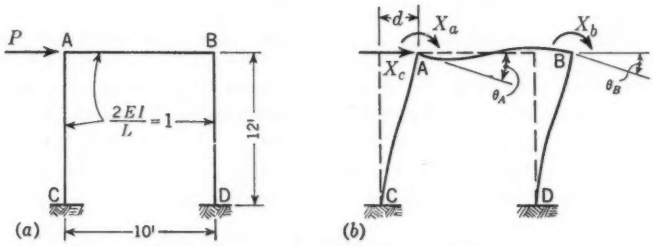

FIG. 4.—ALTERNATE METHOD OF ANALYSIS

By statics, the following three equations may be written:

$$\left.\begin{aligned} +12X_c + M_{AC} + M_{CA} + M_{BD} + M_{DB} &= 0 \\ +X_a - M_{AB} - M_{AC} &= 0 \\ +X_b - M_{BA} - M_{BD} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (37)$$

Expressing end moments by the slope deflection equations, these reduce to

$$\left.\begin{aligned} 12X_c + 3\theta_A + 3\theta_B - d &= 0 \\ 4X_a + 16\theta_A - 4\theta_B + d &= 0 \\ 4X_b - 4\theta_A - 16\theta_B + d &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (38)$$

By Eq. 1, since X_a, X_b, and X_c are characteristic forces, $d = \kappa X_c$; $\theta_A = \kappa X_a$, and $\theta_B = \kappa X_b$, whence

$$\left.\begin{aligned} 3\kappa X_a + 3\kappa X_b + (12 - \kappa) X_c &= 0 \\ (4 - 16\kappa) X_a - 4\kappa X_b + \kappa X_c &= 0 \\ -4\kappa X_a + (4 - 16\kappa) X_b + \kappa X_c &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (39)$$

Eqs. 39 correspond to Eqs. 3, and Eq. 4 becomes

$$\begin{vmatrix} +3\kappa & ; & +3\kappa & ; & +12-\kappa \\ +4-16\kappa; & & -4\kappa & ; & +\kappa \\ -4\kappa & ; & +4-16\kappa; & & +\kappa \end{vmatrix} = 0 \quad \ldots\ldots\ldots (40)$$

the roots to which are: $\kappa_1 = +17.23$; $\kappa_2 = +0.3333$; and $\kappa_3 = +0.1990$. Successively substituting these values of κ into Eqs. 39, the characteristic forces are determined as follows:

$$\left.\begin{aligned} X_{a1} &= +X_{a1}; \quad X_{b1} = +X_{a1}; \quad X_{c1} = +19.77\,X_{a1} \\ X_{a2} &= +X_{a2}; \quad X_{b2} = -X_{a2}; \quad X_{c2} = 0 \\ X_{a3} &= +X_{a3}; \quad X_{b3} = +X_{a3}; \quad X_{c3} = -0.1012\,X_{a3} \end{aligned}\right\} \quad \ldots\ldots\ldots (41)$$

Using Eq. 8 to normalize the characteristic forces,

$$Q_{a1} = \frac{1}{\sqrt{1 + (+1)^2 + (+19.77)^2}} = +0.0505.$$

Likewise, $Q_{a2} = +0.7071$ and $Q_{a3} = +0.7053$. The normalized characteristic forces are, therefore,

$$\left.\begin{array}{lll} Q_{a1} = +0.0505; & Q_{b1} = +0.0505; & Q_{c1} = +0.9975 \\ Q_{a2} = +0.7071; & Q_{b2} = -0.7071; & Q_{c2} = \quad 0 \\ Q_{a3} = +0.7053; & Q_{b3} = +0.7053; & Q_{c3} = -0.0714 \end{array}\right\} \ldots\ldots\ldots (42)$$

The participation factors to be applied to each of these sets of normalized characteristic forces in order that they may correspond to the loading shown in Fig. 4(*a*) are now computed by Eq. 20: $B_1 = P\,Q_{c1} = +0.9975\,P$; $B_2 = P\,Q_{c2} = 0$; and $B_3 = P\,Q_{c3} = -0.0714\,P$. The independent components of distortion for this condition of loading are now computed by Eq. 24:

$$\left.\begin{array}{rl} \theta_A = & +0.9975\,P\,(+17.23)\,(+0.0505) \\ & -0.0714\,P\,(+0.1990)\,(+0.7053) = +0.857\,P \\ \theta_B = & +0.9975\,P\,(+17.23)\,(+0.0505) \\ & -0.0714\,P\,(+0.1990)\,(+0.7053) = +0.857\,P \\ d = & +0.9975\,P\,(+17.23)\,(+0.9975) \\ & -0.0714\,P\,(+0.1990)\,(-0.0714) = +17.14\,P \end{array}\right\} \ldots\ldots\ldots (43)$$

Therefore, $\psi = \dfrac{d}{12} = +1.428\,P$ and the end moments in the frame are as follows:

$$\left.\begin{array}{lll} M_{AB} = 2\theta_A + \theta_B & = 3\,(+0.857\,P) = +2.571\,P & = M_{BA} \\ M_{AC} = 2\theta_A - 3\psi & = 2\,(+0.857\,P) - 3\,(+1.428\,P) = -2.571\,P & = M_{BD} \\ M_{CA} = \quad \theta_A - 3\psi & = +0.857\,P - 3\,(+1.428\,P) = -3.428\,P & = M_{DB} \end{array}\right\} \quad (44)$$

14. Interdependency of Characteristic Forces

For the general case, Eqs. 3 consist of n simultaneous equations with n unknowns. The solution then requires the evaluation of the nth order determinant of Eq. 4 and, for each of the n roots κ thus obtained, the simultaneous solution of $n - 1$ equations of Eqs. 3.

Under certain circumstances, however, the points of application and the directions of characteristic forces may be such that the n equations of Eqs. 3 may break down into two or more groups, each of which may be solved by itself. For example, one might obtain for Eqs. 3, for five forces P_a, P_b, $\cdots$, P_e:

$$P_a\,(\delta_{aa} - \kappa) + P_b\,\delta_{ab} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (45a)$$

$$P_a\,\delta_{ba} + P_b\,(\delta_{bb} - \kappa) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (45b)$$

$$P_c\,(\delta_{cc} - \kappa) + P_d\,\delta_{cd} + P_e\,\delta_{ce} = 0 \ldots\ldots\ldots\ldots (45c)$$

$$P_c\,\delta_{dc} + P_d\,(\delta_{dd} - \kappa) + P_e\,\delta_{de} = 0 \ldots\ldots\ldots\ldots (45d)$$

and

$$P_c\,\delta_{ec} + P_d\,\delta_{ed} + P_e\,(\delta_{ee} - \kappa) = 0 \ldots\ldots\ldots\ldots (45e)$$

This condition would arise, for example, if neither of the forces P_a or P_b caused displacements at points c, d, or e along the lines of action of any of the forces

P_c, P_d, or P_e, respectively. In Eqs. 45 there are two separate groups. Eqs. 45*a* and 45*b* may be solved together, involving only a second-order determinant. From these two equations two sets of characteristic forces are found, for each of which P_a and P_b have values, whereas P_c, P_d, and P_e are zero. Eqs. 45*c*, 45*d*, and 45*e* may then be solved together, involving a third-order determinant. From these three equations three sets of characteristic forces are found, for each of which P_c, P_d, and P_e have values, whereas P_a and P_b are zero.

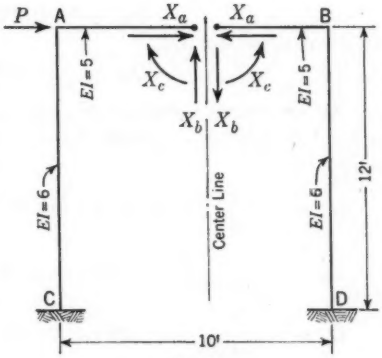

FIG. 5

When Eqs. 3 break down in this manner, a considerable saving in computation is possible, as is illustrated with reference to the structure of Fig. 4(*a*), by taking the redundants as shown in Fig. 5. Values of δ_{mn} may be determined by the usual methods of structural analysis, leading to

$$\left.\begin{array}{lll} \delta_{aa} = +192; & \delta_{ab} = 0; & \delta_{ac} = +24 \\ \delta_{ba} = 0; & \delta_{bb} = +116.6; & \delta_{bc} = 0 \\ \delta_{ca} = +24; & \delta_{cb} = 0; & \delta_{cc} = +6 \end{array}\right\} \quad \ldots\ldots\ldots (46)$$

Eqs. 3 become:

$$P_a(+192-\kappa) + P_c(+24) = 0 \quad \ldots\ldots\ldots\ldots (47a)$$

$$P_b(+116.6-\kappa) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (47b)$$

$$P_a(+24) + P_c(+6-\kappa) = 0 \quad \ldots\ldots\ldots\ldots\ldots (47c)$$

First, solving Eq. 47*b* directly, $\kappa_2 = +116.6$ and $P_{b2} = +P_{b2}$. Next, solving Eqs. 47*a* and 47*c* simultaneously,

$$\begin{vmatrix} +192-\kappa; & +24 \\ +24 & ; +6-\kappa \end{vmatrix} = 0$$

which reduces to $\kappa^2 - 198\kappa + 576 = 0$, from which $\kappa_1 = +195$ and $\kappa_3 = +2.95$. For $\kappa_1 = +195$,

$$\left.\begin{array}{ll} P_{a1} = +P_{a1}; & P_{c1} = +0.127\,P_{a1} \\ & \text{and, for } \kappa_3 = +2.95, \\ P_{a3} = +P_{a3}; & P_{c3} = -7.88\,P_{a3} \end{array}\right\} \quad \ldots\ldots\ldots\ldots (48)$$

By Eq. 8, $Q_{a1} = +0.992 \left(= \dfrac{1}{\sqrt{1+(0.127)^2}} \right)$; $Q_{b2} = +1.000$; and Q_{a3}

$= + 0.126$. Thus the normalized characteristic forces are:

For $\kappa_1 = + 195$,

$$Q_{a1} = + 0.992; \quad Q_{c1} = + 0.126$$

for $\kappa_2 = + 116.6$,

$$Q_{b2} = + 1.000$$

and, for $\kappa_3 = + 6.00$,

$$Q_{a3} = + 0.126; \quad Q_{c3} = - 0.992 \qquad \text{.....(49)}$$

By the usual methods of structural analysis, the following deflections due to load P may be obtained: $\delta_{ao} = + 96\,P$; $\delta_{bo} = - 60\,P$; $\delta_{co} = + 12\,P$. Then, by Eq. 27:

$$B_1 = - \frac{1}{195}[+ 0.992\,(+ 96\,P) + 0.126\,(+ 12\,P)] = - 0.496\,P$$

$$B_2 = - \frac{1}{116.6}[+ 1.000\,(- 60\,P)] = + 0.514\,P \qquad (50)$$

$$B_3 = - \frac{1}{6.00}[+ 0.126\,(+ 96\,P) - 0.992\,(+ 12\,P)] = - 0.063\,P$$

and, by Eqs. 25:

$$X_a = - 0.496\,P\,(+ 0.992) - 0.063\,P\,(+ 0.126) = - 0.500\,P$$
$$X_b = + 0.514\,P\,(+ 1.000) = + 0.514\,P \qquad \text{...(51)}$$
$$X_c = - 0.496\,P\,(+ 0.126) - 0.063\,P\,(- 0.992) = 0$$

The end moments in the members of the frame may now be computed by statics, and will be found to equal those obtained in Section 13.

15. Use of Characteristic Redundants Acting at Elastic Center

In the analysis of a fixed end arch, it is well known that the redundants may be advantageously taken as acting at the elastic center of the arch. The resulting solution may be interpreted as one wherein the redundants are characteristic forces, chosen in such a manner that Eqs. 3 break down into three independent equations. To illustrate the foregoing, consider the structure of Fig. 4(*a*) by taking the redundants as acting at the elastic center as shown in Fig. 6, and applying the procedures developed in this paper. For this structure, the elastic center lies on the vertical axis of symmetry for the structure, and 4.00 ft below the girder AB.

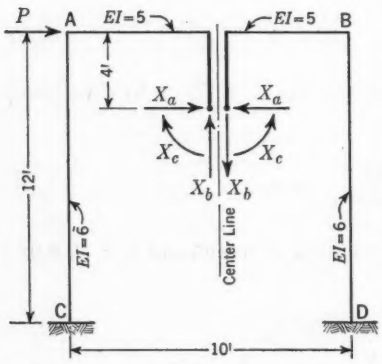

Fig. 6

Values of δ_{mn} may be determined by the usual methods of structural analysis, leading to:

$$\begin{array}{lll} \delta_{aa} = + 96.0; & \delta_{ab} = 0; & \delta_{ac} = 0 \\ \delta_{ba} = 0; & \delta_{bb} = + 116.6; & \delta_{bc} = 0 \\ \delta_{ca} = 0; & \delta_{cb} = 0; & \delta_{cc} = + 6.00 \end{array} \qquad \text{.....(52)}$$

Eqs. 3 become:

$$\left.\begin{aligned} P_a(+96.0-\kappa)&=0\\ P_b(+116.6-\kappa)&=0\\ P_c(+6.00-\kappa)&=0 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

Solving these equations separately:

$$\left.\begin{aligned} \kappa_1&=+116.6; & P_{b1}&=+P_{b1}\\ \kappa_2&=+96.0; & P_{a2}&=+P_{a2}\\ \kappa_3&=+6.00; & P_{c3}&=+P_{c3} \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(54)$$

Application of Eq. 8 leads to the following sets of normalized characteristic loads: $Q_{b1}=\frac{1}{\sqrt{1}}=+1$; $Q_{a2}=\frac{1}{\sqrt{1}}=+1$; and $Q_{c3}=\frac{1}{\sqrt{1}}=+1$. By the usual methods of structural analysis, the following deflections due to load P may be obtained: $\delta_{ao}=+48\,P$; $\delta_{bo}=-60\,P$; and $\delta_{co}=+12\,P$. Then, by Eq. 27:

$$\left.\begin{aligned} B_1&=-\frac{1}{116.6}(+1)(-60\,P)=+0.514\,P\\ B_2&=-\frac{1}{96.0}(+1)(+48\,P)=-0.500\,P\\ B_3&=-\frac{1}{6.00}(+1)(+12\,P)=-2.000\,P \end{aligned}\right\}\quad\ldots\ldots\ldots(55)$$

and, by Eqs. 25,

$$\left.\begin{aligned} X_a&=-0.500\,P(+1)=-0.500\,P\\ X_b&=+0.514\,P(+1)=+0.514\,P\\ X_c&=-2.000\,P(+1)=-2.000\,P \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(56)$$

The end moments in the members of the frame may now be computed by statics, and will be found to equal those obtained in Section 13.

16. Remarks Concerning Solution of Eqs. 3

In Section 2, it was assumed that all values of κ were real and separate. In all solutions made by the writer, values of κ have been real. Should imaginary values of κ be encountered, it is believed that the physical conditions stated by Eqs. 3 are impossible of fulfilment, and that as a result the analysis cannot properly be applied. It is considered doubtful that this condition will arise.

It is possible for two or more values of κ to be equal numerically, however, although this condition is infrequent. Suppose that of the n values of κ, a given value κ' appears m times. It will then be found that only $n-m$ equations of Eqs. 3 are independent when κ' is substituted, so that only $n-m$ values of P can be expressed in terms of the remaining m values of P. This leaves $m-1$ degrees of freedom for each value of κ', so that $m(m-1)$ further conditions must be imposed. These conditions must lead to m separate sets of characteristic forces, which must be mutually orthogonal. Within the limitation of these requirements, arbitrary conditions may be imposed, since

Eqs. 3 will be satisfied. The foregoing may be illustrated by the following example: Consider the equations

$$\left.\begin{aligned}(10.62-\kappa)\,P_a - 2.25\,P_b - 2.25\,P_c + 0.88\,P_d &= 0\\ -2.25\,P_a + (10.62-\kappa)\,P_b + 0.88\,P_c - 2.25\,P_d &= 0\\ -2.25\,P_a + 0.88\,P_b + (10.62-\kappa)\,P_c - 2.25\,P_d &= 0\\ +0.88\,P_a - 2.25\,P_b - 2.25\,P_c + (10.62-\kappa)\,P_d &= 0\end{aligned}\right\}\quad\ldots\ldots(57)$$

The determinant formed by the coefficients to forces P reduces to: $(\kappa - 16.00)(\kappa - 9.74)^2(\kappa - 7.00) = 0$. The roots are: $\kappa_1 = +16.00$; $\kappa_2 = +9.74$; $\kappa_3 = +9.74$; and $\kappa_4 = +7.00$. When the double root $\kappa_2 = \kappa_3 = +9.74$ is substituted into Eqs. 57, only two independent equations are formed, which reduce to

$$P_a = -P_d \quad \text{and} \quad P_b = -P_c \ldots\ldots\ldots\ldots\ldots\ldots(58a)$$

Since Eqs. 58*a* do not define the relations between P_a and P_b, any assumption may be made regarding this relation, and Eqs. 57 will be satisfied. Suppose that for one set of characteristic forces, one such assumption is made arbitrarily, such as $P_b = C\,P_a$, in which C is a numerical factor. This arbitrary assumption defines the following set of characteristic forces:

$$P_a = +P_a;\quad P_b = +C\,P_a;\quad P_c = -C\,P_a;\quad \text{and}\quad P_d = -P_a \ldots(58b)$$

For the second set of characteristic forces, a second assumption such as $P_b = D\,P_a$ may be made, in which D is another numerical factor which cannot, however, be chosen arbitrarily. This defines the second set of characteristic forces as:

$$P_a = +P_a;\quad P_b = +D\,P_a;\quad P_c = -D\,P_a;\quad \text{and}\quad P_d = -P_a \ldots(58c)$$

The two sets of characteristic forces defined by Eqs. 58*b* and 58*c* must satisfy the orthogonal requirements of Eq. 14, whence

$$\begin{aligned}&(+P_a)(+P_a) + (+C\,P_a)(+D\,P_a)\\ &+(-C\,P_a)(-D\,P_a) + (-P_a)(-P_a) = 0 \ldots\ldots\ldots\ldots(59)\end{aligned}$$

and $D = -1/C$. Thus for κ_2, for example, one may choose C as -1 arbitrarily; that is, one may select $P_b = -P_a$, thus defining the first set of characteristic forces as:

$$P_{a2} = +P_{a2};\quad P_{b2} = -P_{a2};\quad P_{c2} = +P_{a2};\quad \text{and}\quad P_{d2} = -P_{a2} \ldots(60a)$$

Then for κ_3, D must be computed from the relation $D = -1/C = -1/-1 = +1$, thus leading to the second set of characteristic forces:

$$P_{a3} = +P_{a3};\quad P_{b3} = +P_{a3};\quad P_{c3} = -P_{a3};\quad \text{and}\quad P_{d3} = -P_{a3} \ldots(60b)$$

Since these two sets of characteristic forces are both associated with the same numerical value of κ, the distinction as to which shall be given the subscript 2 and which the subscript 3 is arbitrary. A different assumption as to the value of C would have led to different sets of characteristic forces, but it is believed that the over-all solution would not be modified thereby.

17. Alternate Basis for Orthogonality

In the treatment to this subject, sets of characteristic forces have been used, which were orthogonal with respect to each other. It is not necessary that the orthogonality exist between sets of forces; an alternate basis for procedure would be to operate on sets of forces so chosen that the qth set of forces is orthogonal with respect to the deflections caused by the rth set of forces. This condition may be illustrated by the equation

$$S_{aq}\,\Delta_{ar} + S_{bq}\,\Delta_{br} + \cdots + S_{nq}\,\Delta_{nr} = 0 \qquad (61)$$

In Eq. 61, the force applied at point m, of the qth set of such forces, is denoted by S_{mq}; Δ_{mr} denotes the total deflection at point m due to the rth set of such forces.

If n such sets of forces, each composed of n forces, are to be determined, there are in all n^2 forces to which values must be assigned. For the first, second, $\cdots$, nth sets of forces one may arbitrarily assign n, $n-1$, $n-2$, $\cdots$, 1 forces, respectively, the only limitation in assigning these values being that each set of forces shall be independent with respect to the other sets. The remaining forces, which consist of 0, 1, $\cdots$, $n-1$ forces for the first, second, $\cdots$, nth sets of forces, respectively, must be computed on the basis of Eq. 61; this will necessitate the respective solution of 0, 1, $\cdots$, $n-1$ sets of simultaneous equations. By the judicious choice of the $\frac{n}{2}(n+1)$ arbitrary values to be assigned (such as by assigning values of + 1 and 0), the amount of computation in arriving at the $\frac{n}{2}(n-1)$ dependent values may be greatly reduced.

In a truss statically indeterminate to the nth degree, let the redundant stresses X_a, X_b, $\cdots$, X_n be expressed in terms of such forces by superposition, as follows:

$$\left.\begin{aligned} X_a &= C_1 S_{a1} + C_2 S_{a2} + \cdots + C_n S_{an} \\ X_b &= C_1 S_{b1} + C_2 S_{b2} + \cdots + C_n S_{bn} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ X_n &= C_1 S_{n1} + C_2 S_{n2} + \cdots + C_n S_{nn} \end{aligned}\right\} \qquad (62)$$

in which C_1, C_2, $\cdots$, C_n are participation factors for the 1st, 2d, $\cdots$, nth sets of forces, respectively. Let the set of forces S_{aq}, S_{bq}, $\cdots$, S_{nq} be denoted by S_q; let the total stress in any bar due to S_q acting on the primary structure be denoted by F_q. Let F_o be the total stress in any bar due to external loads acting on the primary structure.

Suppose the primary structure, acted upon by the external loads and by all the redundants $C_1 S_1$, $C_2 S_2$, $\cdots$, $C_n S_n$, be subjected to a virtual distortion due to an increase in the value of C_m by ΔC_m. Then the redundant $C_m S_m$ will do external virtual work which may be expressed by

$$W_1 = C_m F_{am}(\Delta C_m \Delta_{am}) + C_m F_{bm}(\Delta C_m \Delta_{bm}) + \cdots + C_m F_{nm}(\Delta C_m \Delta_{nm}) = \Delta C_m C_m \sum(F_m \Delta_m) \qquad (63a)$$

The other redundants will do no external virtual work in accordance with Eq. 61. The external loads will do external virtual work, which may be ex-

pressed by

$$W_2 = \Delta C_m \sum\left(\frac{F_o F_m L}{A E}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (63b)$$

Thus the total external virtual work, which equals zero, is given by $W_1 + W_2$, whence

$$C_m = -\frac{\sum\left(\frac{F_o F_m L}{A E}\right)}{\sum(F_m \Delta_m)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (64)$$

Hence if n sets of forces satisfying Eq. 61 can be obtained, Eqs. 64 and 62 lead to the evaluation of $X_a, X_b, \cdots, X_n$. The manner in which these n sets of forces may be obtained, as well as the application of this entire procedure, may be illustrated by a consideration of the structure of Fig. 3(a), for which $\frac{L}{A E} = 25$ for each member. By the usual methods of structural analysis (see Section 12) the following values of δ_{mn} are obtained:

$$\left.\begin{array}{lll} \delta_{aa} = +100; & \delta_{ab} = +16; & \delta_{ac} = 0 \\ \delta_{ba} = +16; & \delta_{bb} = +100; & \delta_{bc} = +16 \\ \delta_{ca} = 0; & \delta_{cb} = +16; & \delta_{cc} = +100 \end{array}\right\} \quad \ldots\ldots\ldots\ldots (65)$$

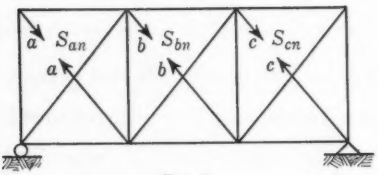

FIG. 7

Referring to Fig. 7, the first set of forces will be arbitrarily chosen as: $S_{a1} = +1$; $S_{b1} = 0$; and $S_{c1} = 0$. Then

$$\left.\begin{array}{l} \Delta_{a1} = +1\,(\delta_{aa}) = +100 \\ \Delta_{b1} = +1\,(\delta_{ba}) = +16 \\ \Delta_{c1} = +1\,(\delta_{ca}) = 0 \end{array}\right\} \quad \ldots\ldots (66)$$

The second set of forces, by inspection, is: $S_{a2} = 0$; $S_{b2} = 0$; and $S_{c2} = +1$. For these forces: $\Delta_{a2} = +1\,(\delta_{ac}) = 0$; $\Delta_{b2} = +1\,(\delta_{bc}) = +16$; and $\Delta_{c2} = +1\,(\delta_{cc}) = +100$ so that Eq. 61 is obviously satisfied. For the third set of forces, $S_{b3} = +1$ is selected arbitrarily but S_{a3} and S_{c3} must be computed so as to satisfy Eq. 61. Considering the first and third sets of forces: $S_{a3}\,(+100) + 1\,(+16) + S_{c3}\,(0) = 0$; and $S_{a3} = -0.160$. Considering the second and third sets of forces: $S_{a3}\,(0) + 1\,(+16) + S_{c3}\,(+100) = 0$; and $S_{c3} = -0.160$. For the S_3 forces:

$$\left.\begin{array}{l} \Delta_{a3} = -0.160\,\delta_{aa} + 1.000\,\delta_{ab} - 0.160\,\delta_{ac} = 0 \\ \Delta_{b3} = -0.160\,\delta_{ba} + 1.000\,\delta_{bb} - 0.160\,\delta_{bc} = +94.88 \\ \Delta_{c3} = -0.160\,\delta_{ca} + 1.000\,\delta_{cb} - 0.160\,\delta_{cc} = 0 \end{array}\right\} \quad \ldots\ldots (67)$$

To summarize, the three sets of forces are:

For S_1,

$$S_{a1} = +1 \quad ; \quad S_{b1} = 0 \; ; \quad S_{c1} = 0$$

for S_2,

$$S_{a2} = 0 \quad ; \quad S_{b2} = 0 \; ; \quad S_{c2} = +1$$

and, for S_3,

$$S_{a3} = -0.160; \quad S_{b3} = +1; \quad S_{c3} = -0.160$$

$$\left.\begin{array}{l} \text{For } S_1, \quad S_{a1} = +1 \; ; \; S_{b1} = 0 \; ; \; S_{c1} = 0 \\ \text{for } S_2, \quad S_{a2} = 0 \; ; \; S_{b2} = 0 \; ; \; S_{c2} = +1 \\ \text{and, for } S_3, \quad S_{a3} = -0.160; \; S_{b3} = +1; \; S_{c3} = -0.160 \end{array}\right\} \quad \ldots\ldots (68)$$

The total deflections due to these forces are, respectively:

$$\left.\begin{array}{lllll} \text{For } S_1, & & & & \\ & \Delta_{a1} = +100; & \Delta_{b1} = +16 \quad ; & \Delta_{c1} = & 0 \\ \text{for } S_2, & & & & \\ & \Delta_{a2} = & 0; \quad \Delta_{b2} = +16 \quad ; & \Delta_{c2} = & +100 \\ \text{and, for } S_3, & & & & \\ & \Delta_{a3} = & 0; \quad \Delta_{b3} = +94.88; & \Delta_{c3} = & 0 \end{array}\right\} \ldots\ldots\ldots (69)$$

Values of $\sum\left(\frac{F_o F_m L}{A E}\right)$ are obtained from Table 3 wherein values of F_o may

TABLE 3.—EVALUATION OF C_m; ALTERNATE BASIS

Bar	F_o	F_1	F_2	F_3	$F_o F_1$	$F_o F_2$	$F_o F_3$
L_0L_1	$+0.500\,P$	-0.600	....	$+0.096$	$-0.300\,P$	....	$+0.048\,P$
L_1L_2	$+0.250\,P$	....	....	-0.600	....	....	$-0.150\,P$
L_2L_3	....	....	-0.600	$+0.096$	....	....	....
U_0U_1	....	-0.600	....	$+0.096$	....	....	....
U_1U_2	$-0.500\,P$	....	....	-0.600	....	....	$+0.300\,P$
U_2U_3	$-0.250\,P$	....	-0.600	$+0.096$	....	$+0.150\,P$	$-0.024\,P$
L_0U_0	....	-0.800	....	$+0.128$	....	....	....
L_1U_1	$+0.667\,P$	-0.800	....	-0.672	$-0.534\,P$	....	$-0.448\,P$
L_2U_2	$-0.333\,P$	....	-0.800	-0.672	....	$+0.267\,P$	$+0.224\,P$
L_3U_3	$-0.333\,P$	....	-0.800	$+0.128$	....	$+0.267\,P$	$-0.043\,P$
U_0L_1	....	$+1.000$	....	-0.160	....	....	....
L_0U_1	$-0.833\,P$	$+1.000$	....	-0.160	$-0.833\,P$	....	$+0.133$
U_1L_2	....	....	....	$+1.000$	....	....	....
L_1U_2	$+0.417\,P$	....	....	$+1.000$	....	....	$+0.417\,P$
U_2L_3	....	....	$+1.000$	-0.160	....	....	....
L_2U_3	$+0.417\,P$	....	$+1.000$	-0.160	....	$+0.417\,P$	$-0.067\,P$
$\Sigma =$	....	....	....	....	$-1.667\,P$	$+1.101\,P$	$+0.390\,P$

be obtained by statics. Values of F_m are also obtainable by statics; they may be obtained by superposition from stresses due to unit values of X_a, X_b, and X_c, which will already be available, since they will have been used (see Section 12) in evaluating δ_{mn}. For substitution into Eq. 64: $\sum\left(\frac{F_o F_1 L}{A E}\right) = -1.667\,P\,(25)$ $= -41.7\,P$; $\sum\left(\frac{F_o F_2 L}{A E}\right) = +1.101\,P\,(25) = +27.5\,P$; and $\sum\left(\frac{F_o F_3 L}{A E}\right)$ $= +0.390\,P\,(25) = +9.75\,P$. Further,

$$\left.\begin{array}{l} \sum(F_1\,\Delta_1) = +1\,(+100) + 0\,(+16) + 0\,(0) = +100 \\ \sum(F_2\,\Delta_2) = +0\,(0) + 0\,(+16) + 1\,(+100) = +100 \\ \sum(F_3\,\Delta_3) = -0.160\,(0) + 1\,(+94.88) - 0.160\,(0) = +94.88 \end{array}\right\} \ldots (70)$$

Hence, by Eq. 64:

$$C_1 = -\frac{-41.7\,P}{+100} = +0.417\,P; \quad C_2 = -\frac{+27.5\,P}{+100} = -0.275\,P;$$

and

$$C_3 = -\frac{+9.75\,P}{+94.88} = -0.103\,P.$$

Finally, by Eqs. 62:

$$\left.\begin{aligned} X_a &= +0.417\,P\,(+1) - 0.275\,P\,(0) - 0.103\,P\,(-0.160) = +0.433\,P \\ X_b &= +0.417\,P\,(0) - 0.275\,P\,(0) - 0.103\,P\,(+1) \quad\quad = -0.103\,P \\ X_c &= +0.417\,P\,(0) - 0.275\,P\,(+1) - 0.103\,P\,(-0.160) = -0.259\,P \end{aligned}\right\} \quad (71)$$

A further condition could have been imposed with respect to each set of S forces:

$$S_{aq}\,\Delta_{aq} + S_{bq}\,\Delta_{bq} + \cdots + S_{nq}\,\Delta_{nq} = +1 \quad\quad (72)$$

This would amount to normalizing each set of forces with respect to the deflections caused by that set of forces. The detail of the solution would then change slightly; if T replaces S to indicate that Eq. 72 has been observed, and if F_m is interpreted as the stress in any bar due to T_m acting on the primary structure, Eq. 63*a* becomes $W_1 = \Delta C_m\,C_m$ and Eq. 63*b* remains unchanged. The denominator $\sum(F_m\,\Delta_m)$ then drops out of Eq. 64. In Eqs. 62, S should now be replaced by T.

This type of solution is possible for any degree of indeterminacy of a structure, although admittedly the labor involved in determining sets of forces which satisfy Eq. 61 would increase rather rapidly with the number of unknowns. Once such sets of forces have been established, however, the redundant stresses may be determined without further recourse to simultaneous equations. This is the same advantage that is gained by use of the elastic center, which, however, is limited in its direct benefits to structures with three or fewer redundants.

18. Conclusion

The purpose of this paper has been to present some of the principles involved in making use of orthogonal forces in the analysis of statically indeterminate structures. Illustrative examples demonstrate these principles, but the structures analyzed were chosen because of their suitability for illustrating principles rather than because their solution by the methods discussed was advantageous.

Structural engineers have been using orthogonal forces for some time in various analyses, but, in some instances, perhaps, without realizing that they were doing so. It is hoped that the presentation of this paper will stimulate further consideration of the possible advantages that may be obtained by their use.

APPENDIX

Notation

The following letter symbols, adopted for use in the paper, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials[3] and American Standard Letter Symbols for

[3] ASA—Z10a—1932.

Mechanics,[4] prepared by committees of the American Standards Association, with Society representation, and approved by the Association in 1932 and 1942, respectively:

A = a factor in Eq. 5; A_{mn} is a factor by which P_{an} is multiplied to obtain P_{mn}; also (see Eq. 30), A = cross-section area of a structural member;
B = a participation factor in Eq. 17; $B_1, B_2, \cdots, B_n$ are participation factors for the first, second, and nth sets of normalized characteristic forces, $R_a, R_b, \cdots, R_n$, respectively;
C = a numerical factor in Eq. 58*b*; also, $C_1, C_2, \cdots, C_n$ are participation factors (Eq. 62) for the sets of forces $S_1, S_2, \cdots, S_n$, respectively;
D = a numerical factor in Eq. 58*c*;
d = lateral distortion of the top of a column (Fig. 4(*b*));
E = modulus of elasticity;
F = total internal stress in a member; thus, F_{nm} is the stress in any bar of the primary structure due to a unit value of Q_{nm}; F_o is the stress in any bar of the primary structure due to external forces only;
I = moment of inertia;
L = length of a structural member;
m = a number of times that the value κ' appears in an equation; also used as a general integer;
n = a number in a series; thus, $1, 2, \cdots, n$; or $a, b, \cdots, n$;
P = an external load (Eq. 2);
Q = a normalized characteristic force; thus, Q_{am} (Eq. 8) represents the complete set of normalized characteristic forces for point a, corresponding to κ_m;
q = a particular set of forces S (Eq. 61);
R = resultant external forces (Eq. 18);
r = a particular set of forces S (Eq. 61);
S = set of forces; S_{mq} is the force applied at point m, of the qth set of such forces;
T = set of forces to replace S in Eq. 72; T_m is a load applied to the primary structure which produces a stress F_m;
W = external virtual work; W_{mn} is the external virtual work done by the forces Q_m when a structure acted upon by one set of forces Q_m is subjected to a virtual distortion corresponding to the deformation that occurs when a second set of forces Q_n is applied to the structure;
X = redundant total stress;
Δ = total deflection; Δ_n is the total deflection at point n in the direction of P_n due to forces $P_a, P_b, \cdots, P_n$;
δ = deflection due to unit load; δ_{mn} is the deflection at any point m in the direction of P_m, due to a unit value of P_n;
θ = angular distortion, the subscript denoting point of rotation;
κ = a proportionality factor; κ' is a given value of κ that occurs m times for every n occurrences of κ;
ψ = a rotation factor in Eq. 44.

[4] ASA—Z10.3—1942.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

TRANSLATORY WAVES IN NATURAL CHANNELS

By J. H. Wilkinson,[1] M. Am. Soc. C. E.

Synopsis

The phenomena of translatory wave travel in natural channels have been studied for many years without entirely satisfactory results. Published studies have been largely theoretical, supplemented by laboratory tests in some instances, but with few observed data on waves in natural channels. Stream flow regulation by storage reservoirs in the Tennessee River Basin during the past few years has developed numerous translatory waves of various magnitudes. Under such conditions, waves can be selected for study which travel considerable distances without being appreciably affected by tributary inflows.

This paper presents data on natural channel translatory waves on the Clinch and the lower Tennessee rivers, and on Wheeler Reservoir in Alabama for a considerable range of flow in each type of channel. Theoretical velocities are computed for these waves by formulas and are compared with observed velocities. Results indicate that wave velocities on the Clinch River can be computed with reasonable accuracy, and that velocities for the rising face of a wave on the lower Tennessee River can be approximated. Computed velocities for waves in Wheeler Reservoir are in poor agreement with observed velocities.

Translatory Waves on the Clinch River

Regulated discharges at Norris Dam in Tennessee have developed a number of well-defined translatory waves which were not affected appreciably by tributary inflow in the reach of the Clinch River between Norris Dam and Wheat stream gage 65.3 miles downstream (see Fig. 1). The Clinch River at Norris Dam drains a mountainous area and has an average annual flow of 4,600 cu ft per sec. Below the dam the river drains a hilly country and has

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **November 1, 1944.**

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an average slope of about 1.6 ft per mile in the reach from the dam to the Wheat gage.

Continuous stage recorders located at intervals throughout this reach provide records from which gage-height graphs in Fig. 2 are plotted. Norris

TABLE 1.—RATINGS, CHANNEL AREAS, AN

Gage height Z (ft)	Norris Gage			Clinton Gage			Edgemoor		
	Discharge Q (cu ft per sec)	Channel area A (sq ft)	Velocity c_1 (miles per hr)	Discharge Q (cu ft per sec)	Channel area A (sq ft)	Velocity c_1 (miles per hr)	Discharge Q (cu ft per sec)	Channel area A (sq ft)	Velocity (miles hr)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	....	420	....	100	1,260	0.9	120	1,350	1.1
1	11	750	....	520	1,500	1.6	870	1,690	2.0
2	790	1,150	2.1	1,120	1,750	1.8	2,030	2,040	2.8
3	2,210	1,550	2.8	1,850	2,010	2.1	3,680	2,400	3.7
4	4,050	1,950	3.5	2,710	2,270	2.4	5,980	2,770	4.5
5	6,310	2,360	4.0	3,690	2,540	2.6	8,500	3,150	4.5
6	9,000	2,790	4.4	4,790	2,830	2.8	11,050	3,550	4.3
7	11,860	3,220	....	6,010	3,130	3.0	13,600	3,950	4.1
8	14,760	3,650	....	7,350	3,440	3.2	16,150	4,360	4.0
9	17,660	4,100	....	8,810	3,750	3.4	18,700	4,780	....
10	20,560	4,550	....	10,390	4,050	3.6	21,250	5,220	3.9
11	23,500	5,000	....	12,090	4,360	3.8	23,800	5,670	3.8
12	26,400	5,450	....	13,910	4,670	4.0	....	....	....
13	29,500	5,900	4.5	15,850	4,990	4.2	....	....	....
14	32,800	6,350	....	17,910	5,310	4.4	....	....	....
15	35,900	6,800	....	20,090	5,640	4.6	....	....	....
16	....	....	....	22,390	5,980	4.7	....	....	....
17	....	....	....	24,810	6,320	....	....	....	....
18	....	....	....	27,350	6,680	4.8	....	....	....

[a] Gage heights at Norris, see Fig. 1.

and Wheat gages have been rated by the U. S. Geological Survey, and ratings for Clinton and Edgemoor gages are developed from these ratings. Discharge ratings and corresponding channel cross-section areas for these stations are

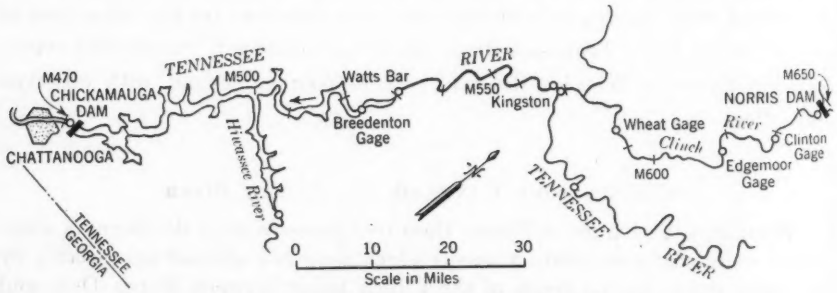


FIG. 1.—CLINCH RIVER AND UPPER TENNESSEE RIVER

given in Table 1. The gage-height relation between Wheat gage and Gallahers Ferry gage is also given. As discharge measurements for rating Wheat gage are made at Gallahers Ferry, data on the channel cross section at Wheat gage are not available. Pertinent data at these gages are given in Table 2.

Velocity of a Translatory Wave.—To determine the velocity of a translatory wave from observed data, it is necessary to identify identical points on the wave at successive stations. In this study (1) the beginning-of-rise and (2) the midpoint-of-rise are used for this purpose on rising waves, and (3) the

VELOCITIES, CLINCH RIVER STATIONS (SEE FIG. 1)

Gage		Gallahers Ferry			Norris Gage to Wheat Gage[a]		
Velocity c_1 (miles per hr)	Wheat gage discharge Q (cu ft per sec)	Relative gage height Z (ft)	Channel area A (sq ft)	Wheat gage velocity c_1 (miles per hr)	Wave velocity c_1 (miles per hr)	Water velocity[b] V (miles per hr)	Gage height Z (ft)
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(1)
1.1	....	....	....	....	....	....	0
2.0	....	....	....	....	....	....	1
2.8	0	2.05	1,030	....	2.0	0.4	2
3.7	590	3.52	1,460	1.5	2.7	0.8	3
4.5	1,780	4.93	1,890	2.2	3.2	1.2	4
4.5	3,300	6.27	2,310	2.6	3.6	1.5	5
4.3	4,900	7.58	2,730	2.9	3.8	1.8	6
4.1	6,610	8.75	3,110	3.1	3.9	2.1	7
4.0	8,360	9.88	3,500	3.2	4.0	2.3	8
....	10,160	11.01	3,880	3.4	4.1	2.5	9
3.9	12,000	12.07	4,250	3.5	4.15	2.7	10
3.8	13,940	13.16	4,630	....	4.2	2.8	11
....	15,940	14.21	5,010	3.6	....	....	12
....	17,940	15.25	5,380	....	....	....	13
....	20,040	16.28	5,770	3.7	....	....	14
....	22,140	17.31	6,150	....	....	....	15
....	24,340	18.30	6,530	3.8	....	....	16
....	....	....	....	....	....	....	17
....	....	....	....	....	....	....	18

[b] V = velocity, mean (Q/A).

beginning-of-fall and (4) the midpoint-of-fall on falling waves. Velocities indicated by the points of fully developed rise and fully developed fall were investigated but were found to be unsatisfactory, partly because these points could not be identified accurately and partly because these points are modified considerably by other influences.

The following formulas[2] are used for computing theoretical velocities from channel characteristics:

$$c_1 = \frac{0.682\,dQ}{b\,dZ} \quad \ldots (1a)$$

and

$$c_2 = \frac{0.682\,(Q_2 - Q_1)}{A_2 - A_1} \quad \ldots (1b)$$

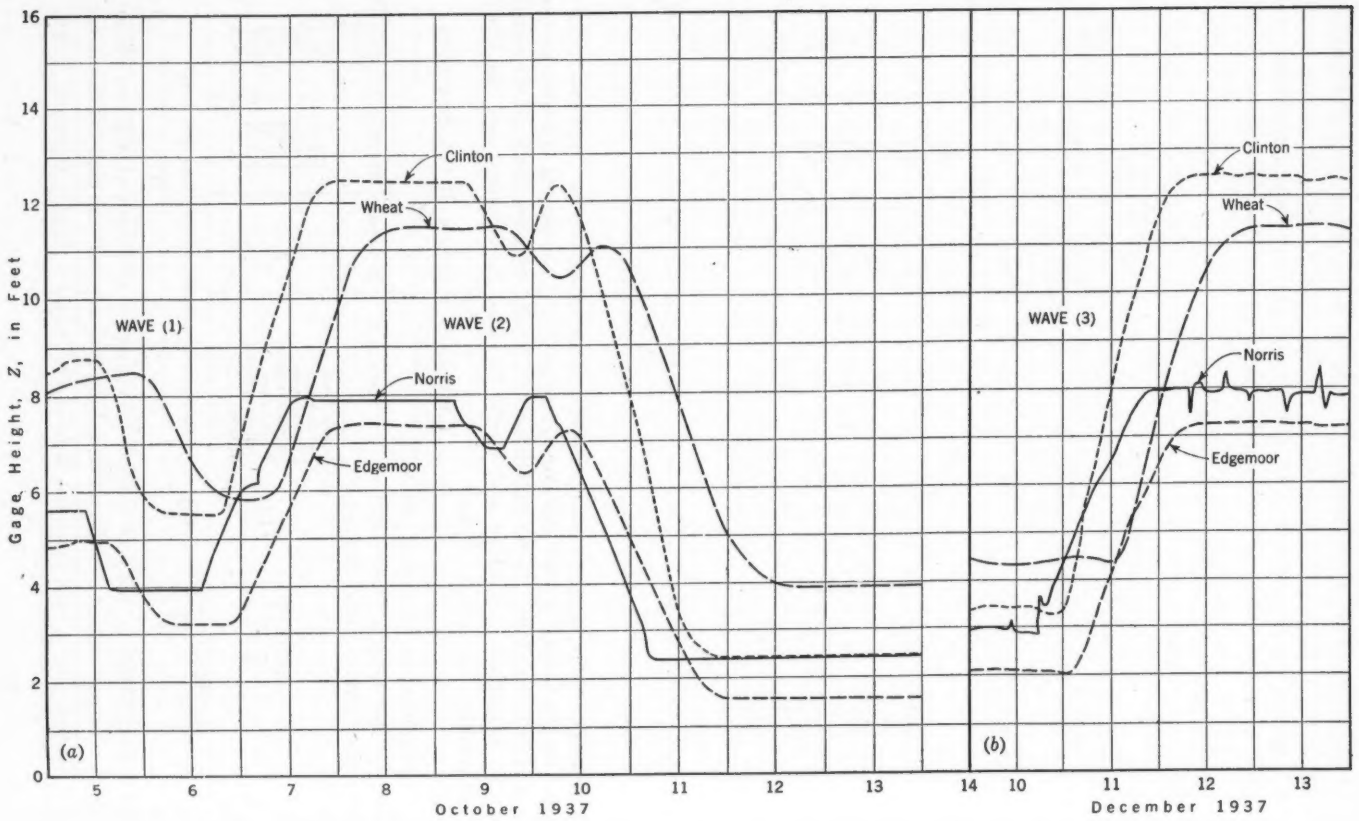
TABLE 2.—CLINCH RIVER STATIONS

Station	River mile	Elevation[a]	Drainage area (sq miles)
Norris Dam	79.8	....	2,912
Norris gage	78.7	819.11	2,913
Clinton gage	59.0	778.54	3,056
Edgemoor gage	47.6	768.08	3,089
Wheat gage	14.5	717.40	3,385
Gallahers Ferry	13.2	713.53	3,387
Mouth	0	....	4,413

[a] Elevation of gage zero above Mean Gulf Level.

in which c_1 is the velocity of a wave traveling on a stream at a given stage; c_2 is the average velocity of the wave, taking into account the channel discharge

[2] "The Hydraulics of Flood Movements in Rivers," by Harold A. Thomas, *Engineering Bulletin*, Carnegie Inst. of Tech., 1934, pp. 18–19.


Gage Height, Z, in Feet
16
14
12
10
8
6
4
2
0
WAVE (1)
WAVE (2)
WAVE (3)
Clinton
Wheat
Norris
Edgemoor
(a)
(b)
5
6
7
8
9
10
11
12
13
14
October 1937
December 1937

20
90
10
Clinton

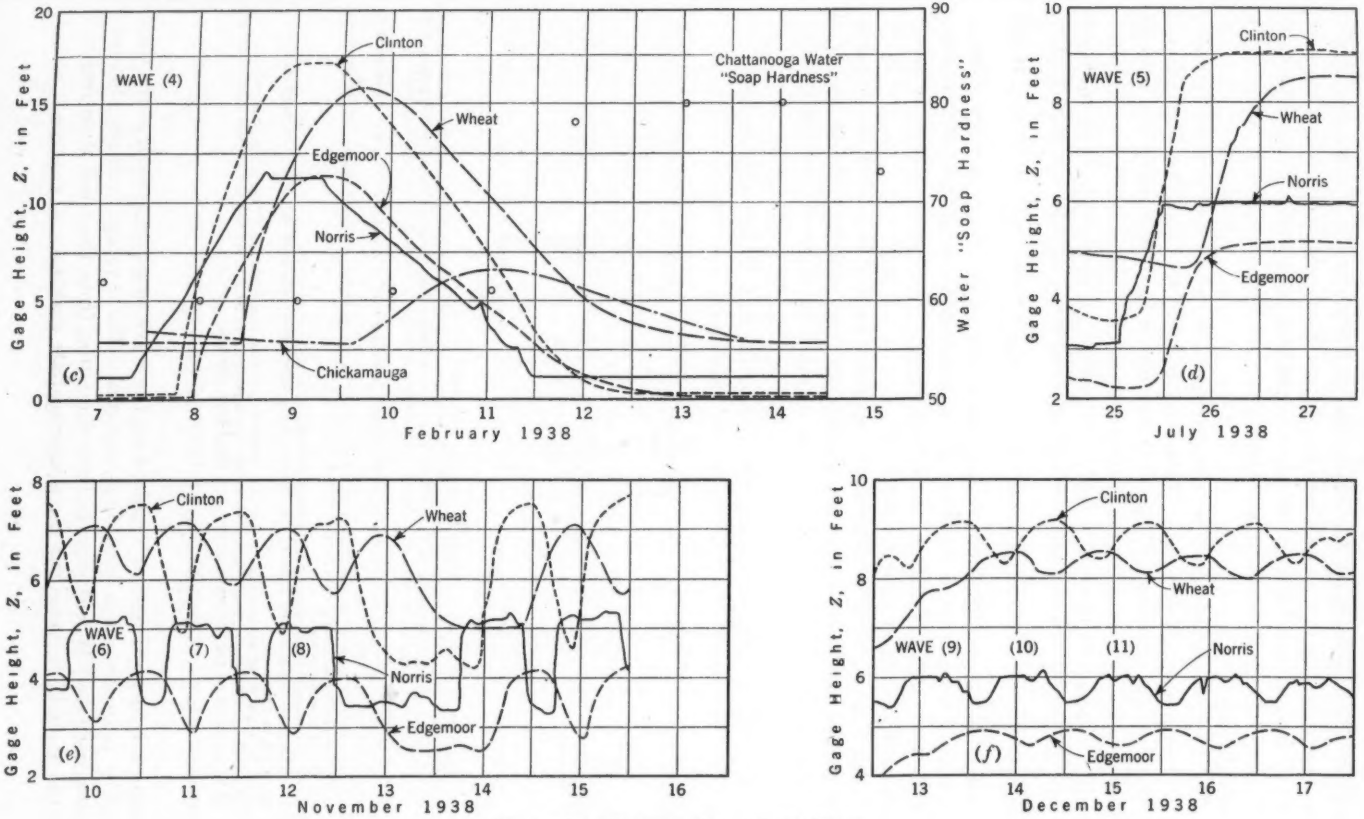


FIG. 2.—GAGE HEIGHTS, CLINCH RIVER WAVES

characteristics throughout the entire rise of the wave; Q_1 and Q_2 are the discharges corresponding to gage heights Z_1 and Z_2, respectively; dQ is an increment of discharge; dZ is a selected increment in a gage height Z; b is the width of the channel; and A_1 and A_2, respectively, are the cross-section areas below stages Z_1 and Z_2.

Eq. 1*a* is applicable only for determining the velocity of a wave traveling on a stream at a given stage. This becomes evident when the derivation of the equation is examined. Eq. 1*b* is applicable for determining the average velocity of the wave, as it takes into account the channel discharge characteristics throughout the entire rise of the wave. Results from both equations will be the same when the discharge characteristic of the channel, that is, $\frac{dQ}{b\,dZ}$, is constant throughout the range in stage covered by the wave. It will be noted that c_1 is a function of the slope of the rating curve at a given stage, whereas c_2 is a function of the slope of a secant to the rating curve between two stages, or the average slope of the rating curve. The constant 0.682 in Eqs. 1 is inserted to convert to miles per hour.

Using Eqs. 1, wave velocities are computed for the four identifiable points on each wave at each of the four stage recording stations. These computed velocities are assumed to exist half the distance to adjacent stations in determining the theoretical time required for the wave to traverse each of the three reaches between stage recording stations. The average velocity between Norris gage and Wheat gage is then computed from the total time required for the wave to traverse the three intermediate reaches.

TABLE 3.—COMPUTATION OF c_1 (EQ. 1*a*) FOR BEGINNING-OF-RISE

Station	Observed Z_1	b (ft)	dQ (cu ft per sec)	dZ (ft)	c_1 (Eq. 1*a*)	Reach, L (miles)	Time, t (hr)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Norris	4.0	417	210	0.1	3.43	9.85	2.87
Clinton	5.5	282	110	0.1	2.66	15.55	5.85
Edgemoor	3.2	369	210	0.1	3.88	22.25	5.72
Wheat	5.8	....	139	0.1	2.92	16.55	5.67
Gallahers	7.3	323	....	....	....	....	....
Average c_1 (Norris to Wheat)					3.19	64.2	20.11

Col. 1.—Stations at which data on gage heights are available.
Col. 2.—The river stage at the beginning-of-rise, Z_1, is taken from the gage-height graph for each station in Fig. 2(*a*). As data on the channel at Wheat are not available, the stage at Gallahers corresponding to the observed stage is determined from the gage-height relation in Table 1 for computing the velocity at this station.
Col. 3.—Channel widths b at these stages are determined from the rate of change in channel cross-section area at stage Z_1 from Table 1.
Col. 4.—The rates of change in discharges, dQ, per 0.1 ft change in stage, at these stages are determined from the discharge rating curves in Table 1.
Col. 5.—0.1 ft change in stage.
Col. 6.—c_1 is computed by substituting values in Cols. 3, 4, and 5 in Eq. 1*a*.
Col. 7.—The reach length L in miles is the distance over which the computed velocity c_1 at each station is assumed to exist.
Col. 8.—The number of hours required for the wave to traverse this distance if traveling at a velocity c_1. The average velocity c_1, from Norris to Wheat, is then equal to 64.2 miles divided by the total number of hours required for the wave to traverse these four reaches.

The method of applying Eqs. 1 is illustrated in Tables 3 and 4, using data from Wave 1, October 6, 1937.

As the velocity c_1 in Eq. 1*a* at a given station is a function of the stage, curves for values of c_1 can be developed for each station as shown in Fig. 3. A curve for the average velocity c_1 for the Norris-to-Wheat reach can also be

TABLE 4.—COMPUTATION OF c_2 (EQ. 1*b*) FOR THE MIDPOINT-OF-RISE

Station	Z_2	Z_1	Q_2	Q_1	A_2	A_1	Q_2-Q_1	A_2-A_1	c_2 (Eq. 1*b*)	L (miles)	t (hr)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Norris	7.8	4.0	14,180	4,050	3,560	1,950	10,130	1,610	4.30	9.85	2.28
Clinton	12.5	5.5	14,860	4,210	4,830	2,680	10,650	2,150	3.38	15.55	4.61
Edgemoor	7.4	3.2	14,620	4,080	4,110	2,470	10,540	1,640	4.38	22.25	5.08
Wheat	11.3	5.8	....	....	....	....	....	....	3.24	16.55	5.12
Gallahers	13.5	7.3	14,540	4,590	4,736	2,642	9,950	2,094	....	....	....
Average c_2 (Norris to Wheat)									3.75	64.2	17.09

Col. 1.—Stations at which data on gage heights are available.
Col. 2.—Gage height Z_2 at end-of-rise from Fig. 2(*a*) (observed).
Col. 3.—Gage height Z_1 at beginning-of-rise (observed).
Col. 4.—Discharge Q_2 corresponding to gage height Z_2 determined from Table 1.
Col. 5.—Discharge Q_1 corresponding to gage height Z_1.
Col. 6.—Channel cross-section area A_2, corresponding to gage height Z_2 determined from Table 1.
Col. 7.—Area A_1 corresponding to gage height Z_1.
Col. 8.—Q_2-Q_1 from Cols. 4 and 5.
Col. 9.—A_2-A_1 from Cols. 6 and 7.
Col. 10.—Wave velocity c_2 computed by substituting Cols. 8 and 9 in Eq. 1*b*.
Cols. 11 and 12.—c_2, L, and t (Cols. 10, 11, and 12) are determined in the same manner as corresponding values in Table 3.

developed by determining stages at successive stations corresponding to a selected stage at Norris gage by mean gage-height relations and by computing the average velocity c_1. Such a curve indicates average values of c_1 plotted against Norris stages for the Norris-to-Wheat reach under average conditions of local inflow. It gives the same value of c_1 as computed in Table 3. This occurs only when average conditions of local inflow exist and the stages at all stations agree with the mean gage-height relation used for constructing this curve.

For the purpose of comparison, a curve showing the water velocity in the Norris-to-Wheat reach is also shown in Fig. 3 (water velocity is the velocity indicated by a current meter). This curve is developed from the discharge rating and channel cross-section areas at each station for stages determined by gage-height relations, assuming that each station velocity exists halfway to adjacent stations. The water velocity is about half the wave velocity at ordinary stages and less than half at low stages.

In the computation of c_2 by Eq. 1*b*, in Table 4, an inconsistency in discharges Q_2 and Q_1 at successive stations will be noted. This can be caused by either an error in the station ratings, errors in observed stages, or irregularities in flow. It is believed that this inconsistency will not affect results materially as long as the area difference, $A_2 - A_1$, corresponds to the discharge difference, $Q_2 - Q_1$.

Theoretical velocities c_1 and c_2 for the beginning and midpoint of rising waves, and for the beginning and midpoint of falling waves shown in Fig. 2,

together with the velocity c_o, corresponding to the observed time of travel from Norris to Wheat, are shown in Table 5.

Comparison of Observed and Computed Wave Velocities.—Fig. 4 shows velocities c_o, c_1, and c_2 plotted against the gage heights observed at Norris gage, with mean curves drawn for each. Mean curves of computed velocities for the

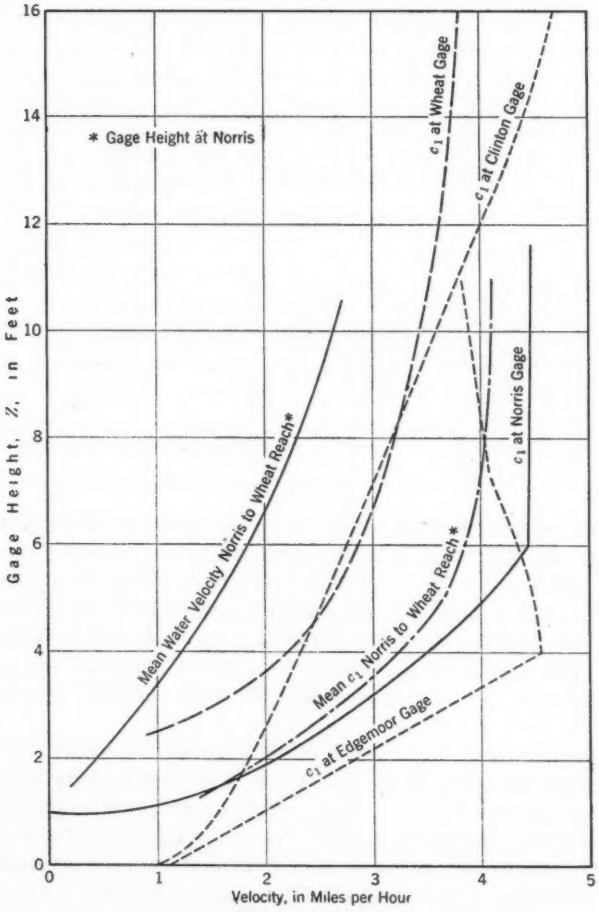


FIG. 3.—VELOCITY CURVES, NORRIS DAM TO WHEAT GAGE

midpoint-of-rise (Fig. 4(*b*)) and the midpoint-of-fall (Fig. 4(*d*)) agree very well with the mean curves for observed velocities. It will be noted that the range of gage heights covered by these data is small, and, the gage heights being midpoints, do not extend to either extremely low or high stages. Results for the beginning-of-rise (Fig. 4(*a*)) and the beginning-of-fall (Fig. 4(*c*)) do not agree nearly as well, computed velocities being smaller than the observed in both cases. In the latter plottings, velocities c_2 are computed from the total

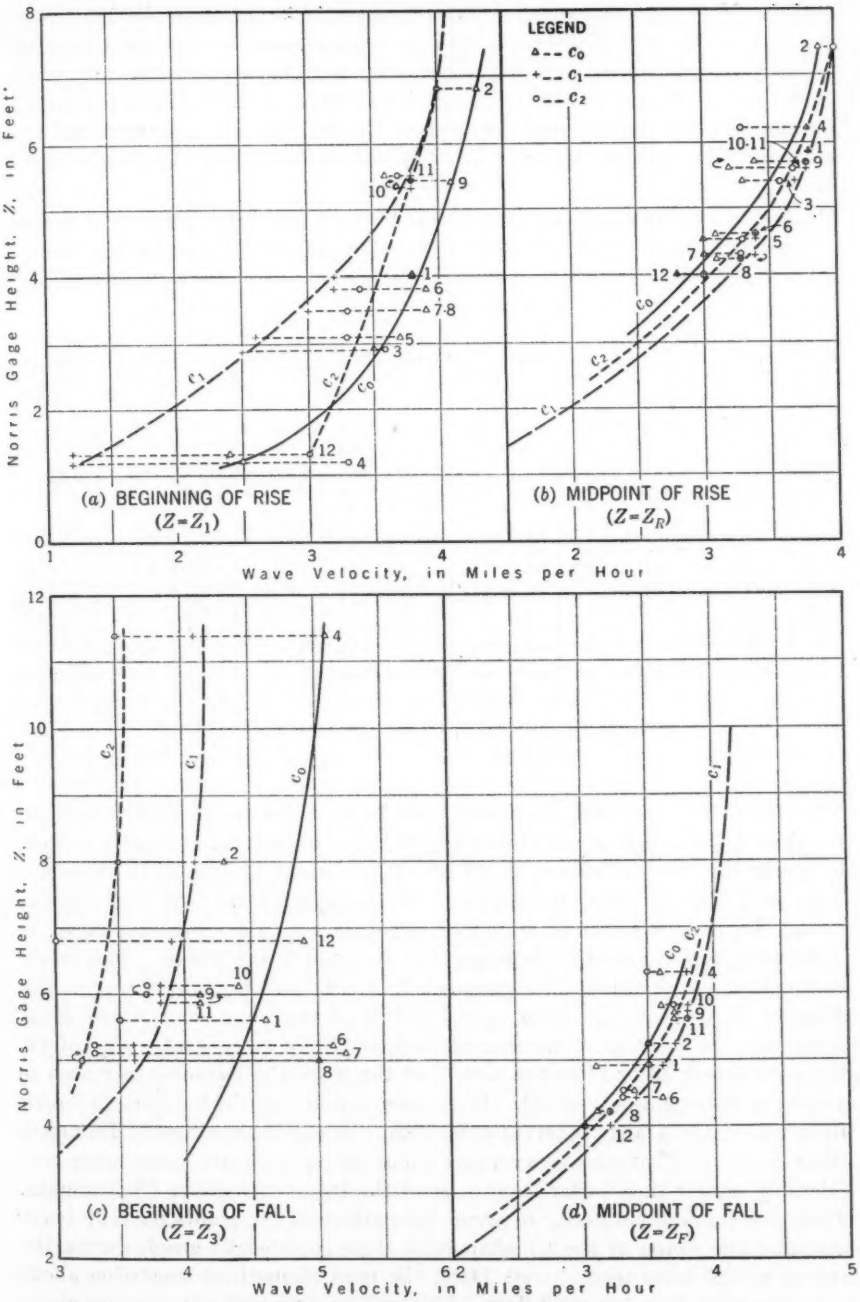


FIG. 4.—VELOCITY, CLINCH RIVER WAVES

rise or fall of the wave and are plotted against Z_1 or Z_2, respectively. Mean curves for observed velocities for both the beginning-of-rise and the beginning-of-fall are higher at a given stage than observed velocities for the midpoints.

Wave of February 7, 1938.—The wave of February 7, 1938 (Wave 4, Table 5), developed when the quantity released at Norris Dam was increased during a comparatively low and steady flow in the Clinch and upper Tennessee rivers.

TABLE 5.—OBSERVED AND COMPUTED WAVE VELOCITIES BETWEEN NORRIS AND WHEAT GAGES, CLINCH RIVER WAVES, IN MILES PER HOUR

Wave No.	Date	Rising Waves								Falling Waves							
		Norris		Beginning-of-Rise[a]		Norris	Midpoint-of-Rise			Norris		Beginning-of-Fall		Norris	Midpoint-of-Fall		
		Z_1	Z_2	c_o	c_1	Z_R	c_o	c_1	c_2	Z_1	Z_2	c_o	c_1	Z_F	c_o	c_1	c_2
	1937																
1	Oct. 6	4.0	7.8	3.8	3.2	5.9	3.8	3.8	3.8	5.6	4.0	4.6	3.8	4.9	3.1	3.6	3.5
2	Oct. 9	6.8	7.9	4.3	4.0	7.4	3.9	4.0	4.0	8.0	2.4	4.3	4.0	5.2	3.5	3.7	3.5
3	Dec. 10	2.9	8.0	3.5	2.5	5.4	3.3	3.7	3.6	...	...	...	...	...	...	...	...
	1938																
4	Feb. 7	1.2	11.3	2.5	1.2	6.2	3.8	3.8	3.3	11.4	1.2	5.1	4.1	6.3	3.6	3.8	3.5
5	July 25	3.1	5.9	3.7	2.6	4.5	3.0	3.4	3.3	...	...	...	...	...	...	...	...
6	Nov. 10	3.8	5.2	3.9	3.2	4.5	3.1	3.4	3.4	5.2	3.5	5.1	3.6	4.4	3.6	3.4	3.3
7	Nov. 11	3.5	5.1	3.9	3.0	4.3	3.0	3.4	3.3	5.1	3.6	5.2	3.6	4.4	3.4	3.4	3.3
8	Nov. 12	3.5	5.1	3.9	3.0	4.3	3.1	3.3	3.3	5.1	3.4	5.0	3.6	4.2	3.1	3.2	3.2
9	Dec. 13	5.4	6.0	4.1	3.8	5.7	3.4	3.8	3.8	6.0	5.4	4.1	3.8	5.7	3.7	3.8	3.7
10	Dec. 14	5.4	6.0	3.7	3.8	5.7	3.2	3.8	3.7	6.0	5.5	4.4	3.8	5.8	3.6	3.8	3.7
11	Dec. 15	5.5	6.0	3.6	3.8	5.7	3.2	3.8	3.7	6.0	5.4	4.1	3.8	5.7	3.7	3.8	3.8
	1940																
12	Feb. 8	1.2	6.7	2.4	1.2	4.0	2.8	2.8	3.0	6.8	1.2	4.9	3.9	4.0	3.0	3.2	3.0

[a] c_o is the wave velocity corresponding to the observed time of travel.

The release was gradually increased from 30 cu ft per sec to 24,300 cu ft per sec in 33 hr, was held about steady for 14 hr, and then was gradually reduced to about the original release in 48 hr. Gage-height graphs of the resulting wave at Clinch River stations and at Chickamauga Dam site, near Chattanooga, Tenn., are shown in Fig. 2(*c*), together with the "soap hardness" of water samples from the Chattanooga City Pumping Plant intake. This intake is about six miles below Chickamauga Dam site and the water travels this distance in about 3 hr. Prior to the arrival of the wave from Norris Dam, soap-hardness tests at Chattanooga showed a fairly constant value of 60. However about 48 hr after the arrival of the wave the hardness increased to a value of 80 for a 2-day period. It has been found that the hardness of Norris Reservoir water is fairly constant at 95 ppm, whereas the hardness of Tennessee River water at Chattanooga averages about 65 ppm, Norris water being considerably harder than water from most of the tributaries above Chattanooga. There is a marked similarity between the pattern of the points for soap hardness and the shape of the Chickamauga Dam gage-height graph during the arrival of the wave from Norris Dam, the peak of hardness occurring about two days later than the peak flow. It is unfortunate that data are not avail-

able on the hardness of the water in the several tributaries coincident with the passage of the wave.

If the hardness peak indicates the arrival of actual water released from Norris Dam, it requires four days, corresponding to a velocity of about 2 miles per hr, for water to travel 183.3 miles from Norris Dam to Chattanooga, compared with 53 hr (3.5 miles per hr) for the wave. The flow in the river at Chattanooga before the arrival of the wave was 23,000 cu ft per sec. This would indicate that the actual water in the wave when it arrived at Chickamauga Dam site was in the river in front of the original wave developed at Norris Dam.

Translatory Waves in Wheeler Reservoir

Translatory waves in Wheeler Reservoir (Fig. 5) have quite different characteristics from those in tributary streams, such as the Clinch River. The

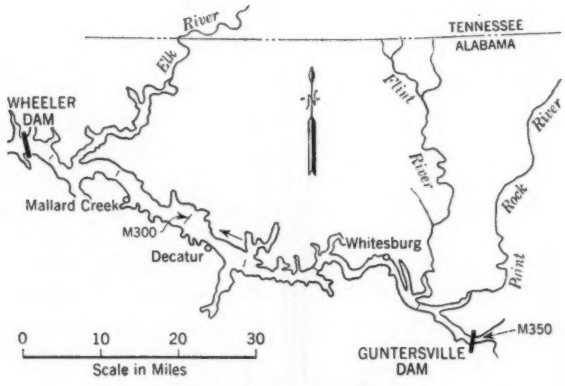


Fig. 5.—Wheeler Reservoir

channel has greater depth and width, together with a flatter water-surface slope. Even with abrupt changes of considerable magnitude in the rate of inflow, translatory waves are quickly smoothed out.

Continuous stage recorders located in Alabama at Guntersville lower lock gage (mile 349.0), Whitesburg (mile 333.7), Decatur (mile 305.1), and Mallard Creek (mile 293.4), together with hourly observations at Wheeler Dam (mile 274.9), provide data from which elevation graphs in Figs. 6 and 7 were prepared. The accompanying hydrographs are prepared from computed discharges at Guntersville and Wheeler dams. The drainage area above Guntersville Dam is 24,450 sq miles with a mean annual discharge of 41,400 cu ft per sec, whereas the drainage area above Wheeler Dam is 29,590 sq miles with a mean annual discharge of 50,000 cu ft per sec.

Theoretical Formulas.—Changes in discharge at Guntersville Dam, which developed these waves, were made quickly enough to be considered as sudden for the purpose of analysis. H. W. King,[3] M. Am. Soc. C. E., has given a

[3] "Handbook of Hydraulics," by H. W. King, 3d Ed., 1939, pp. 420–421.

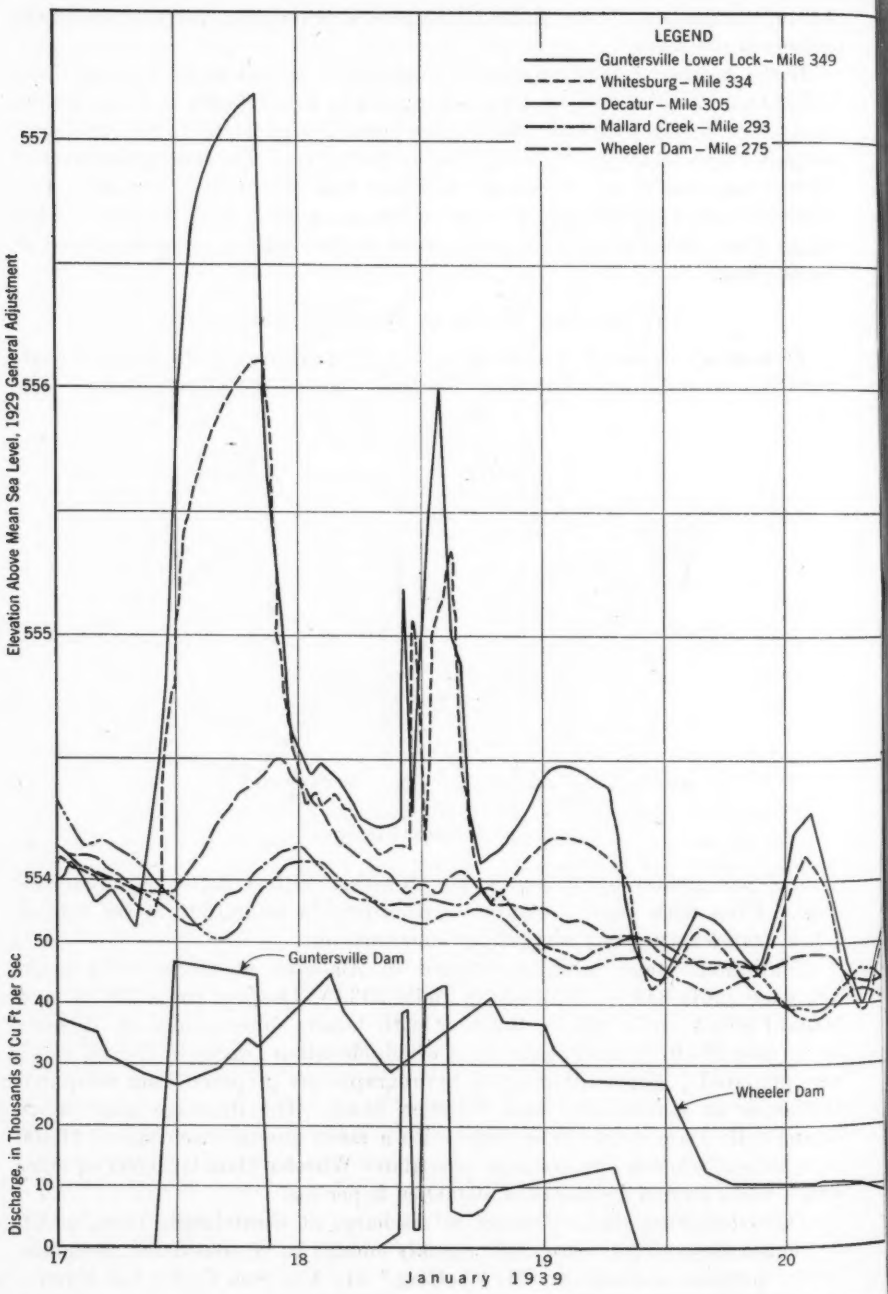


FIG. 6.—GAGE HEIGHTS AND DISCHARGE,

formula that is applicable for computing velocities of such waves:

$$c_3 = \left[V_1 + \sqrt{g \frac{A_2 \bar{y}_2 - A_1 \bar{y}_1}{A_1 \left(1 - \frac{A_1}{A_2}\right)}} \right] 0.682 \quad \ldots\ldots\ldots\ldots (2a)$$

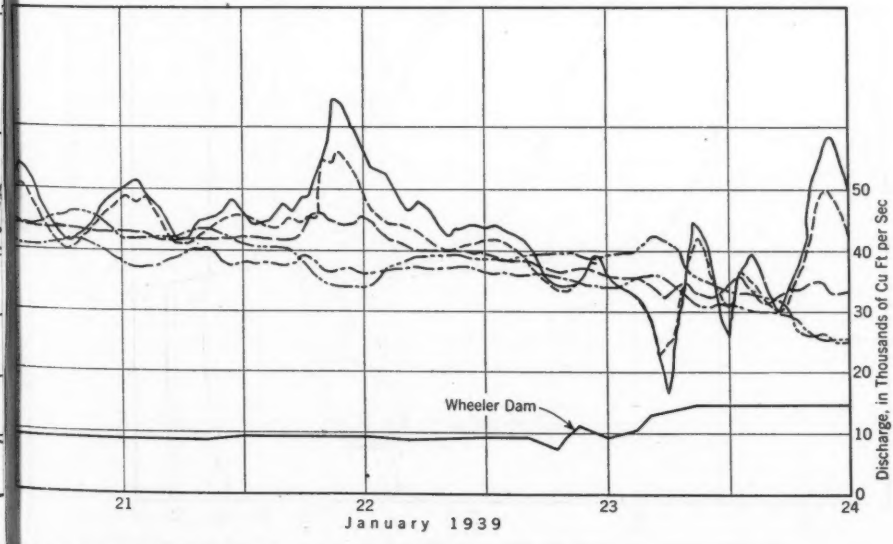
An approximation of this formula, based on the assumption that the difference between A_1 and A_2 is negligible, and also that the channel is of uniform cross section, is given by Professor King[3] as

$$c_4 = [V_1 + \sqrt{g\, y_m}]\, 0.682 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

in which c_3 is the velocity of a wave created suddenly, the effect of the difference between A_1 and A_2 being taken into consideration; V_1 is initial water velocity; g is the acceleration of gravity; $\bar{y}_1$ and $\bar{y}_2$ are the depths to the center of pressure of A_1 and A_2, respectively; A_1 and A_2 are channel cross-section areas occupied by the wave at beginning and end, respectively; c_4 is the velocity of a wave created suddenly, the difference between A_1 and A_2 being considered negligible; and y_m is the mean depth of water. The constant 0.682 is inserted to convert to miles per hour.

The times of travel of observed translatory waves in reservoirs are determined from the beginning-of-rise in rising waves and the beginning-of-fall in falling waves. Observed velocities, c_o, and velocities c_3 and c_4, computed by Eqs. 2, are shown in Table 6 for each of four rising and four falling waves, for each of the four reaches between intermediate gages, and also for the overall reach between Guntersville and Wheeler dams.

Velocities of Translatory Waves in Wheeler Reservoir.—Velocities computed by Eqs. 2 (see Table 6) agree fairly well with the observed velocities in some



WHEELER RESERVOIR WAVES IN JANUARY, 1939

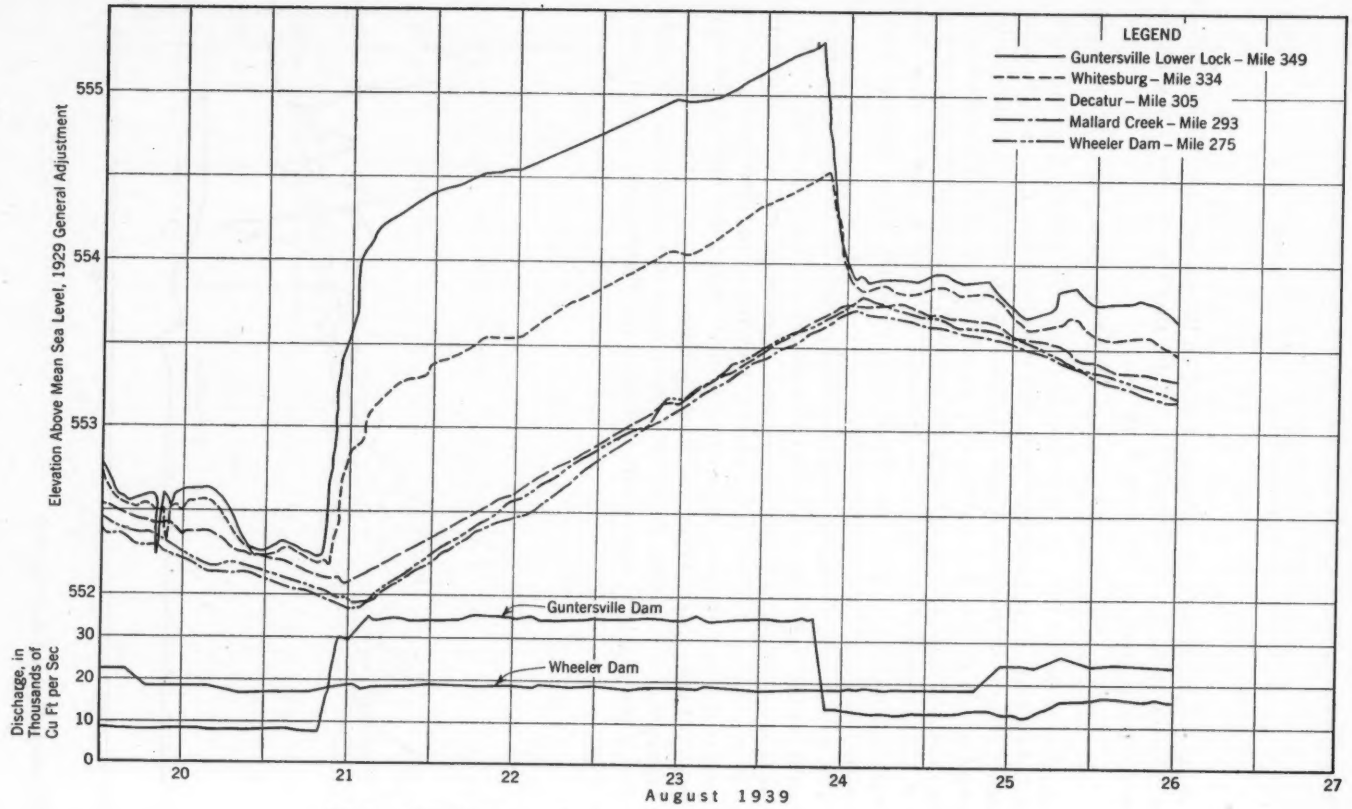


FIG. 7.—GAGE HEIGHTS AND DISCHARGE, WHEELER RESERVOIR WAVES IN AUGUST, 1939

cases and differ widely in others. For rising waves in the Guntersville Dam to Whitesburg reach, computed velocities by Eq. 2*a* and Eq. 2*b* both agree closely with the observed velocities except for the wave of January 17. Observed velocities for this wave are lower in all reaches than observed velocities for the other three waves. In this reach computed velocities c_3 and c_4 are practically the same for all waves.

For rising waves in the Whitesburg to Decatur reach, computed velocities c_4 are about the same as for the previous reach, whereas computed velocities c_3 are consistently lower. The average of velocities, c_3, is practically the same as the average of c_o. For rising waves in the Decatur to Mallard Creek reach, both computed velocities give about the same results, which are higher than observed velocities in all cases. The average of c_o is only a little more than half the observed velocity c_o for the previous reach. For rising waves in the Mallard Creek to Wheeler Dam reach, both computed velocities again give about the same results, but are about double the average of the observed velocities. In the over-all reach from Guntersville Dam to Wheeler Dam, rising wave computed velocities c_3 average slightly lower than computed velocities c_4, but are about one third larger than the observed velocity. Observed and computed velocities for falling waves are similar to those for rising waves.

FIG. 8.—LOWER TENNESSEE RIVER

Considerable difficulty is encountered in determining the effective average channel cross section for the application of Eqs. 2 in the Whitesburg-Decatur and the Decatur-Mallard Creek reaches. Channel cross sections

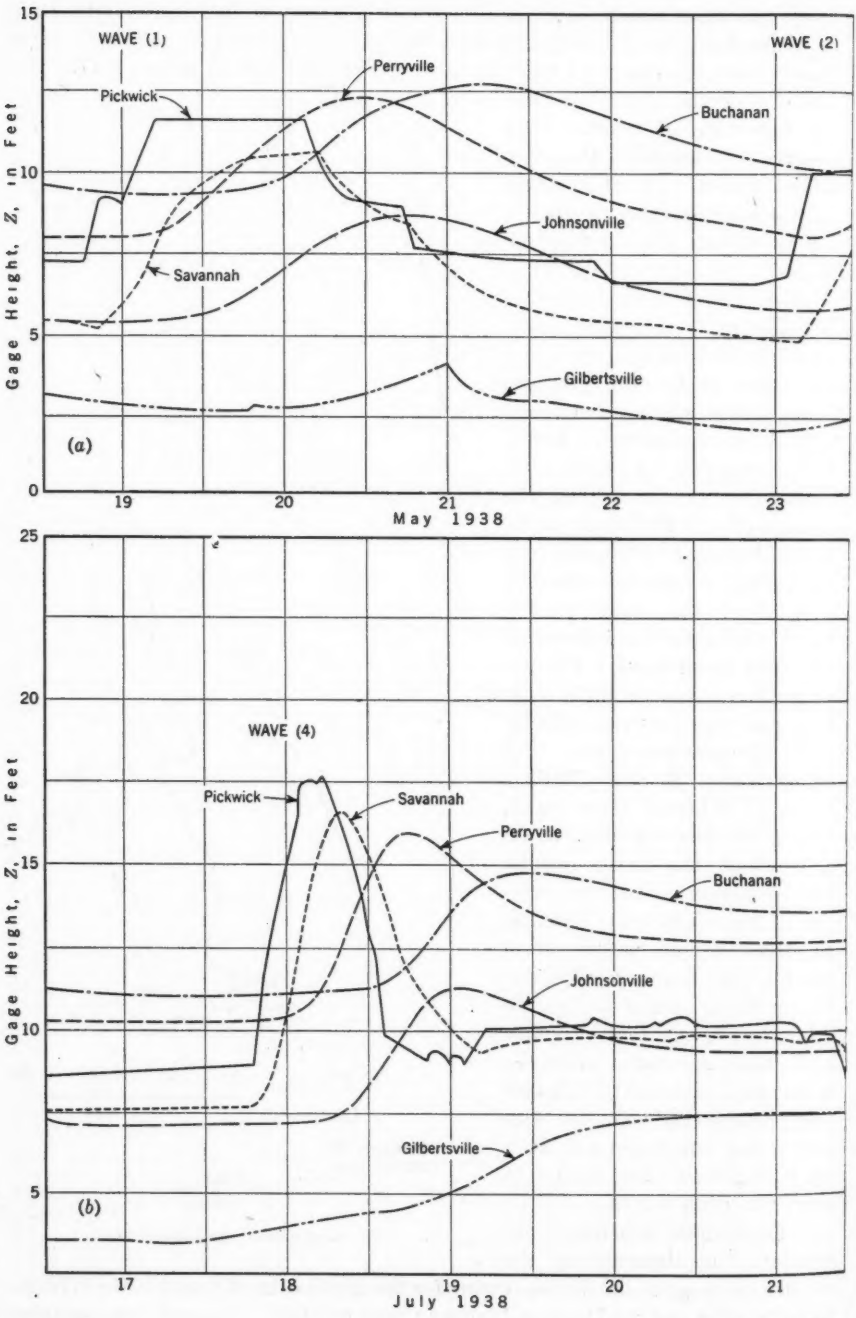
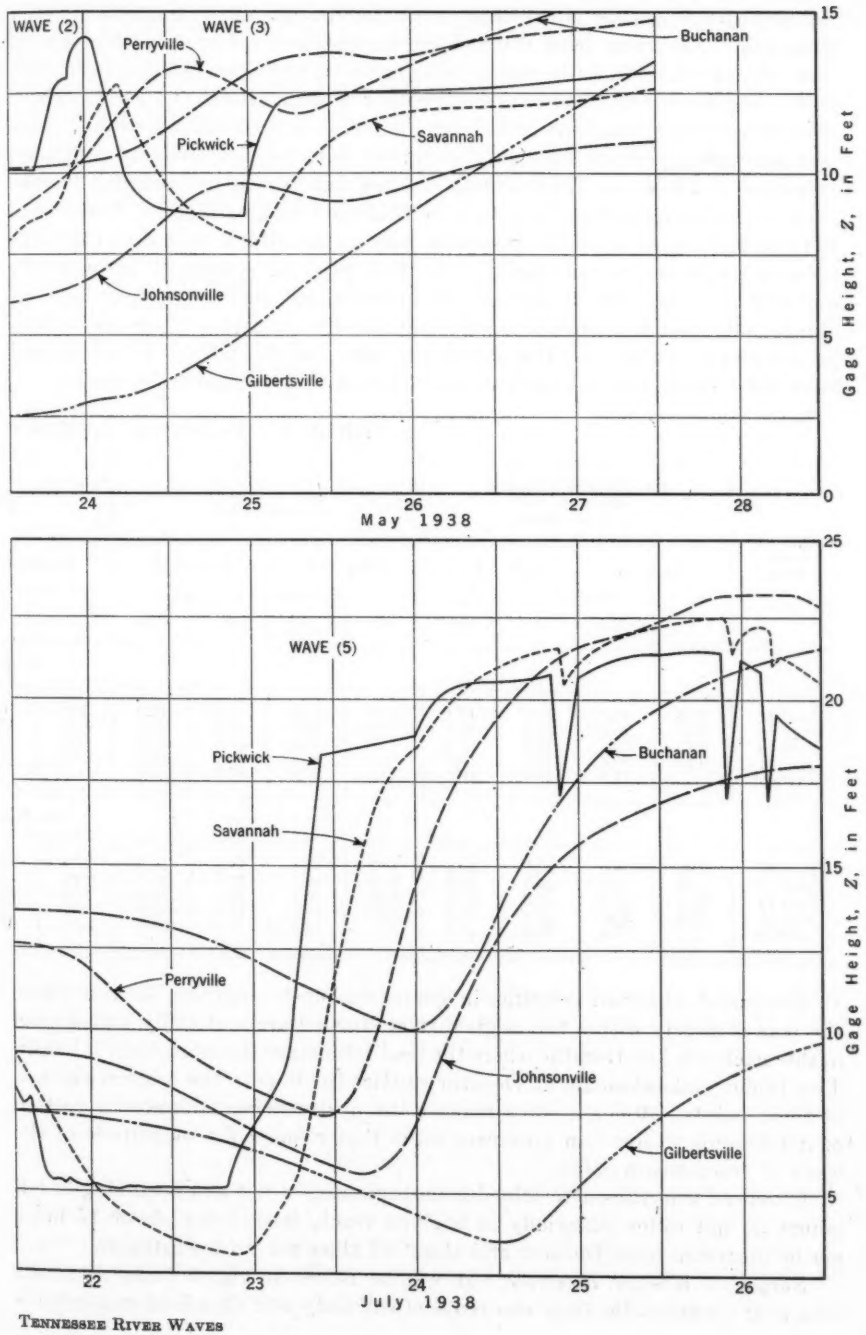


FIG. 9.—GAGE HEIGHTS.



TENNESSEE RIVER WAVES

taken at about one-mile intervals are available throughout the length of Wheeler Reservoir. Upstream from Whitesburg the reservoir is largely within the old river channel and has fairly regular cross sections, while downstream from Mallard Creek the reservoir covers the entire river bottom. Between these two gages there is a gradual transition from river channel to river bottom sections. This may account in part for the wide discrepancy between observed and computed velocities in these two intermediate reaches, but would not account for the even wider discrepancy in the results for Mallard Creek to Wheeler Dam reach. It is probable that when the reservoir widens, the lateral spread of the wave reduces its downstream velocity. Another probable source of error is the inability to determine discharges at intermediate stations. Discharges at Guntersville and Wheeler dams are determined from gate settings and should be reasonably accurate. Discharges at intermediate stations are estimated from these discharges taking into account local inflow from tributaries.

TABLE 6.—VELOCITIES IN MILES PER

Date of wave, 1939	GUNTERSVILLE DAM TO WHITESBURG L = 15 MILES				WHITESBURG TO DECATUR L = 29 MILES				DECATUR TO L = 12	
	Observed		Computed		Observed		Computed		Observed	
	t	c_0	c_2	c_4	t	c_0	c_2	c_4	t	c_0
										(*a*) RISING
Jan. 17	1.50	10.0	19.7	19.1	2.50	11.6	17.6	19.9	2.00	6.0
May 2	0.75	20.0	20.8	20.4	1.50	19.3	17.4	20.9	1.00	12.0
June 19	0.75	20.0	19.0	19.1	1.25	23.2	16.8	20.1	1.25	9.6
Aug. 21	0.75	20.0	19.4	18.6	1.75	16.6	18.3	19.5	0.92	13.0
Average	....	17.5	19.7	19.3	....	17.7	17.5	20.1	....	10.2
										(*b*) FALLING
Jan. 18	0.83	18.0	20.7	20.2	1.92	15.1	18.3	20.5	1.25	9.6
May 3	1.00	15.0	20.1	20.7	1.50	19.3	17.1	20.8	1.00	12.0
June 19	0.83	18.0	21.3	20.0	3.17	9.2	19.1	20.3	1.00	12.0
Aug. 24	0.83	18.0	19.1	19.7	2.67	10.9	17.9	20.2	1.00	12.0
Average	....	17.2	20.3	20.2	....	13.6	18.1	20.4	....	11.4

In general, observed velocities in the reaches upstream from Decatur where the pool is largely within the original river channel are materially higher than in the reaches below Decatur where the pool submerges the original river banks. Two bridge embankments at Decatur restrict the flow to the original channel at these points. When a wave reaches the upstream embankment a portion of it rebounds to form an upstream wave that reduces the magnitude of the wave at downstream points.

Observed velocities determined from beginning-of-rise and beginning-of-fall points do not differ materially in a given reach, both being about 17 miles per hr upstream from Decatur and about 12 miles per hr downstream.

Surges in Wheeler Reservoir.—It will be noted in Fig. 6 that, when the release at Guntersville Dam was reduced suddenly and then held steady for a

time, a series of surges occurred at fairly regular intervals of about 11 hr. This interval is about the time required for a wave to travel from Guntersville Dam to Wheeler Dam and return. Usually the peaks at Guntersville Dam coincide with valleys at Wheeler Dam. These surges have a much smaller magnitude at Decatur than at stations upstream or downstream. Water-surface profiles of the reservoir plotted for elevations observed at successive hours show that the reservoir water surface appears to pivot on Decatur. The water surface falls at upstream stations when it is rising at downstream stations.

Translatory Waves on the Tennessee River Below Pickwick Landing Dam

Translatory waves on the Tennessee River below Pickwick Landing Dam (see Fig. 8) are typical of waves developed by regulated releases in a natural

Hour, Waves in Wheeler Reservoir

Mallard Creek Miles		Mallard Creek to Wheeler Dam L = 18 Miles				Guntersville Dam to Wheeler Dam L = 74 Miles				Date of wave, 1939
Computed		Observed		Computed		Observed		Computed		
c_3	c_4	t	c_0	c_3	c_4	t	c_0	c_3	c_4	
Waves										
13.4	14.0	2.00	9.0	24.7	23.0	8.00	9.3	18.4	19.1	Jan. 17
17.2	14.6	1.50	12.0	23.8	23.6	4.75	15.6	19.2	19.9	May 2
14.7	14.3	1.00	18.0	21.6	23.3	4.25	17.4	17.7	19.3	June 19
13.8	13.4	1.83	9.8	23.5	22.7	5.25	14.1	18.5	18.5	Aug. 21
14.8	14.1	....	12.2	23.4	23.2	....	14.1	18.5	19.2	Average
Waves										
13.9	14.3	1.50	12.0	27.2	23.2	5.50	13.5	19.4	19.6	Jan. 18
18.0	14.9	1.50	12.0	20.2	23.5	5.00	14.8	20.4	20.0	May 3
....	14.3	1.00	18.0	16.4	23.1	6.00	12.3	....	19.5	June 19
16.6	14.0	1.25	14.4	25.3	23.0	5.75	12.9	19.3	19.3	Aug. 24
16.2	14.4	....	14.1	22.3	23.2	....	13.4	19.7	19.6	Average

river channel having a considerable discharge capacity, and a comparatively flat slope. The mean annual discharge of this section of the river is about 60,000 cu ft per sec, and the fall is 65 ft in 207 miles, 0.3 ft per mile.

Waves for which gage-height graphs are shown in Fig. 9 are selected to cover a considerable range of flow conditions and have discontinuities that can be identified at successive downstream stations. Pertinent data on lower Tennessee River gaging stations are given in Table 7. Gage-height graphs are shown for Pickwick lower lock (Tenn.), Savannah (Tenn.), Perryville (Tenn.), Johnsonville (Tenn.), Buchanan (Tenn.), and Gilbertsville (Ky.) gages whenever the records for these stations are available for the wave selected. Stages at Gilbertsville are affected by Ohio River stages at all times, and high stages on the Ohio River affect stages as far upstream as Johnsonville.

Wave Velocities.—Wave velocities for the reach from Savannah to Buchanan are computed by the same formulas as are used for the Clinch River waves. These results, together with observed velocities determined from the beginning-of-rise, midpoint-of-rise, and beginning-of-fall points, are listed in Table 8.

TABLE 7.—LOWER TENNESSEE RIVER STATIONS

Station (see Fig. 8)	River mile	Elevation[a]	Drainage area (sq miles)
Pickwick lower lock	206.4	350.0	32,820
Savannah	189.9	341.61	33,140
Perryville	134.9	325.27	34,500
Johnsonville (near)	100.5	319.61	38,520
Buchanan	66.2	303.0	39,720
Gilbertsville	21.7	299.56	40,200
Ohio River	0	....	40,910

[a] Elevation of gage zero above Mean Gulf Level.

Discharge ratings, area, and channel width curves for each station are given in Table 9. The Pickwick lower lock to Savannah reach was excluded from the velocity comparison because the channel and discharge character at neither of these stations are representative of the intervening reach, and the Buchanan to Gilbertsville reach was excluded because points on very few waves can be identified satisfactorily at Gilbertsville due to the effect of the Ohio River. The method of applying formulas for wave velocities is similar to that previously described for waves on the Clinch River.

Computed wave velocities c_1 and c_2 for the beginning-of-rise point are generally lower than the observed velocity c_o. On the average c_o is about 1.5 times c_1 and 1.4 times c_2. However, the variation is too large to be considered as a consistent relation. For the midpoint-of-rise, computed velocities in general agree much better with the observed. There is also better agreement between c_1 and c_2 than in computed velocities for the beginning-of-rise points. For beginning-of-fall points, the agreement between computed and observed velocities is about the same as for the midpoint-of-rise.

A study of velocities of midpoint-of-fall shows no agreement between observed and computed velocities, the computed velocities being much higher than the observed. When a falling wave travels down the lower Tennessee River, its shape is gradually modified until it finally becomes a recession that follows the characteristic curve of water draining out of the channel.

Factors Affecting Results.—During periods of rapidly changing stage, such as exist during the passage of a wave, the slope of the water surface and the discharge differ considerably from the steady flow conditions. Due to the flat slopes that normally exist on the lower Tennessee River, these variations have a relatively greater effect on the discharge than on the Clinch River which has a steeper slope. This factor accounts in part for results being less satisfactory for the lower Tennessee River than for the Clinch River, and would be most pronounced on the beginning-of-rise where the poorest agreement was found.

Rising waves on the lower Tennessee River have a tendency to flatten and lengthen at successive downstream stations in contrast to the tendency of Clinch River rising waves to maintain their height and to shorten. Falling waves in both rivers have a tendency to recede on a characteristic pattern which depends on the volume of the river channel.

The wide variation and lack of consistency in observed velocities are due partly to inability to identify selected points on the wave with any degree of

TABLE 8.—TRANSLATORY WAVES IN THE TENNESSEE RIVER FROM SAVANNAH (MILE 189.9) TO BUCHANAN (MILE 66.2)

WAVE		SAVANNAH		WAVE VELOCITY			BUCHANAN	
No.	Date	Z_1	Z_2	c_0	c_1	c_2	Z_1	Z_2
	(*a*) VELOCITIES FOR BEGINNING-OF-RISE POINTS							
	1938							
1	May 19	5.2	10.6	6.0	3.9	4.2	9.5	12.8
2	May 23	4.8	12.5	6.1	4.0	4.5	10.3	13.8
3	May 25	7.7	11.7	5.4	4.2	4.6	13.6	15.5
4	July 18	7.7	16.4	6.8	4.3	4.8	11.3	15.0
5	July 23	3.1	22.2	4.5	3.7	4.9	10.0	21.5
6	Dec. 16	4.1	6.7	6.4	3.7	3.9	8.0	9.0
	1939							
7	Jan. 11	9.1	12.9	7.2	4.6	5.1	13.4	14.2
8	Jan. 13	6.2	16.4	7.2	4.5	5.1	14.5	22.1
	(*b*) VELOCITIES FOR MIDPOINTS-OF-RISE							
	1938							
1	May 19	5.2	10.6	4.4	4.4	4.2	9.5	12.8
5	July 23	3.1	22.2	4.3	5.0	4.9	10.0	21.5
6	Dec. 16	4.1	6.7	4.2	3.8	3.9	8.0	9.0
	1939							
8	Jan. 13	6.2	16.4	4.2	5.0	5.1	14.5	22.1
	(*c*) VELOCITIES FOR BEGINNING-OF-FALL POINTS							
	1938							
1	May 20	10.6	4.8	5.1	4.6	4.3	12.8	10.3
2	May 24	12.5	7.8	3.9	4.8	4.9	13.8	13.6
4	July 18	16.3	9.4	4.5	5.0	5.1	15.0	13.8
6	Dec. 17	6.7	4.1	3.9	4.0	3.9	9.0	8.1
	1939							
7	Jan. 11	12.9	6.2	5.1	4.8	5.0	14.2	14.0
8	Jan. 16	17.4	10.4	8.7	5.4	5.2	22.1	17.0

certainty, particularly at the two downstream stations. The midpoint-of-rise can be determined with the most certainty when the wave consists of one continuous rise; but unfortunately this occurs in very few waves.

CHARACTERISTICS OF TRANSLATORY WAVES

When a rising wave is developed by a fairly uniform increase in discharge, as is the case in most of the Clinch River waves, the rising face of the wave tends to retain the initial rate of increase in discharge at successive downstream stations. When the wave develops on an empty channel, as in Fig. 2(*c*), the beginning-of-rise continues to be abrupt at successive stations; but

when the wave develops on a considerable initial flow, as in Fig. 2(*a*), a fillet develops at the beginning-of-rise which enlarges as the wave travels downstream.

A similar fillet occurs as the wave approaches fully developed rise. As the wave progresses downstream these two fillets continue to enlarge until the upper limb of the beginning-of-rise fillet meets the downward limb of the end-of-rise fillet. Although these fillets are enlarging, the duration of rise at the

TABLE 9.—RATINGS, CHANNEL AREAS, AND VELOCITIES

Gage height Z (ft)	PICKWICK			SAVANNAH			PERRYVILLE	
	Discharge Q (cu ft per sec)	Channel area A (sq ft)	c_1 (miles per hr)	Discharge Q (cu ft per sec)	Channel area A (sq ft)	c_1 (miles per hr)	Discharge Q (cu ft per sec)	Channel area A (sq ft)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
2	6,000	3,620	2.2	14,000	10,200	3.4	....	....
4	12,800	5,800	2.5	21,000	11,500	3.8	11,900	18,000
6	21,500	8,000	2.8	28,750	12,700	4.1	18,200	19,200
8	31,500	10,300	3.1	37,500	14,000	4.5	25,600	20,400
10	42,400	12,500	3.3	46,500	15,300	4.8	34,200	21,800
12	54,400	14,910	3.6	55,500	16,700	4.9	43,600	23,000
14	66,400	17,200	3.8	65,500	18,100	5.0	54,000	24,300
16	78,400	19,570	3.9	75,500	19,500	5.0	64,700	25,700
18	90,600	21,900	4.0	86,000	21,000	5.0	76,000	27,000
20	102,800	24,290	4.1	97,000	22,400	5.0	87,600	28,400
22	115,200	26,800	4.2	108,000	23,800	4.9	100,000	29,800
24	127,600	29,060	4.3	119,000	25,300	4.9	113,000	31,200

initial rate becomes shorter and finally disappears. On the Clinch River (see Fig. 1) this seems to occur at some point above Wheat, 65 miles below Norris, and on the lower Tennessee (see Fig. 8) at a point above Perryville, 71 miles below Pickwick.

In the case of a falling wave, a characteristic recession tends to develop regardless of the initial rate of reduction in discharge. This recession is retained at successive stations on the Clinch River until fillets at the beginning-of-fall and the end-of-fall enlarge to cover the entire falling face of the wave. This again appears to occur by the time that waves reach Wheat or Perryville.

When the period of maximum discharge at the crest of a wave continues for only a short period, as in Fig. 2(*c*), the spreading fillets at the end-of-rise and at the beginning-of-fall gradually shorten this period as the wave progresses downstream. Eventually the two fillets will overlap and the discharge will begin to fall off before the full rise has been developed. When this occurs flattening of the wave begins and the magnitude of rise becomes progressively smaller. A valley between two rises is affected in a similar manner, resulting in a gradual reduction in the depth of the valley as is shown in Figs. 2(*e*) and 2(*f*).

CONCLUSIONS

1. Velocities of translatory waves, both rising and falling, can be approximated with either Eq. 1*a* or Eq. 1*b* in natural channels such as the Clinch and

lower Tennessee rivers when rated stations at representative sections are located at suitable intervals throughout the reach.

2. Midpoints of rise or fall are best suited for determining the velocity of an observed wave.

3. Eqs. 2*a* and 2*b* give similar results for velocities of translatory waves in Wheeler Reservoir, but these results are consistently higher than observed velocities.

LOWER TENNESSEE RIVER STATIONS (SEE FIG. 8)

	JOHNSONVILLE			BUCHANAN			
c_1 (miles per hr)	Discharge Q (cu ft per sec)	Channel area A (sq ft)	c_1 (miles per hr)	Discharge Q (cu ft per sec)	Channel area A (sq ft)	c_1 (miles per hr)	Gage height Z (ft)
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(1)
....	12,500	11,000	2.7	....	....	....	2
3.4	21,250	13,200	3.2	7,000	16,300	....	4
4.0	31,750	15,200	3.6	14,500	18,200	2.8	6
4.5	44,000	17,300	3.9	23,500	20,000	3.2	8
4.9	57,000	19,400	4.2	33,000	21,900	3.7	10
5.2	70,000	21,600	4.4	44,500	23,800	4.0	12
5.4	84,000	23,800	4.5	56,500	25,800	4.3	14
5.6	98,000	25,900	4.7	69,500	27,800	4.6	16
5.8	112,000	28,100	4.9	83,000	29,900	4.7	18
6.0	126,000	30,200	5.0	96,000	31,800	4.9	20
6.2	142,000	32,500	5.1	112,000	33,900	5.0	22
6.3	158,000	34,800	5.1	127,000	36,000	4.9	24

4. Within the range of these data the volume of flow appears to have little effect on the velocity of translatory waves in a reservoir.

ACKNOWLEDGMENTS

Acknowledgment for assistance in the preparation of this paper is made to A. J. Cooper, Jr., and T. J. Nordenson, Juniors, Am. Soc. C. E., as well as to the other members of the River Forecasting Section, Hydraulic Data Division, Tennessee Valley Authority.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

WEDGE-SHAPED STRUCTURAL MEMBERS UNDER DIRECT STRESS AND BENDING

By Robert B. B. Moorman,[1] Assoc. M. Am. Soc. C. E.

Synopsis

Presented in this paper are expressions for computing compressive and shearing unit stresses in the concrete, tensile unit stresses in the reinforcement, and bond unit stresses between the reinforcement and concrete for wedge-shaped reinforced concrete members subjected to direct stress and bending. Curves to expedite the use of the expressions are also presented. Only approximate solutions are offered since the materials do not follow strictly the assumptions made in the derivations.

Introduction

The subject of stresses in wedge-shaped reinforced concrete members subjected to direct stress and bending has been treated in relatively few papers. E. Mörsch[2] presented a derivation pertaining to the shearing unit stress for the case in which the compressive surface is normal to the section on which the stress is considered acting, and the reinforcement inclined to that section. Mörsch's treatment serves as a basis for the more extensive derivations of expressions for shearing unit stress and bond unit stress found in this paper. V. Dašek[3] presented expressions with which to compute the shearing unit stress at any point along a section of the member, especially on the compressive side of the neutral axis. However, the shearing unit stress as used in design is a measure of diagonal tension, and for this reason only the stress on the tensile side of the neutral axis is of interest. On the compressive side of the neutral axis the actual shearing unit stress (maximum) would be found to be equal to one half the maximum compressive unit stress and would be acting on planes inclined at 45° to the compressive surface.

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **November 1, 1944.**

[1] Structures Consultant, Goodyear Aircraft Corp., Akron, Ohio.

[2] "Der Eisenbetonbau," by E. Mörsch, Konrad Wittwer, 1929, Vol. 1, Pt. 2, p. 290.

[3] "Die Schubspannungen bei aussermittiger Beanspruchung gerader Stäbe mit veränderlichem Querschnitt," by V. Dašek, *Beton und Eisen*, Vol. 37, 1938, p. 137.

Notation.—The symbols used in this paper are defined where they first appear, either by text or diagram. They conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials[4] and American Standard Letter Symbols for Mechanics,[5] prepared by committees of the American Standards Association, with Society representation, and approved by the Association in 1932 and 1942, respectively.

Assumptions.—The usual assumptions that are made in deriving the ordinary expressions in reinforced concrete design are:

(1) A section plane before bending remains plane after bending;
(2) The reinforcement resists all the tension;
(3) The unit stress in the concrete is proportional to the distance from the neutral axis and in the reinforcement it is n times the proportionality for the concrete, in which n is the ratio between modulus of elasticity of steel and modulus of elasticity of concrete; and
(4) Perfect adhesion exists between the concrete and the reinforcement.

Bending Stresses

Fig. 1 shows part of a wedge-shaped reinforced concrete member acted upon by an external force, R. This force can be resolved into components N (normal) and V (shear) as shown. The total compression on the section is

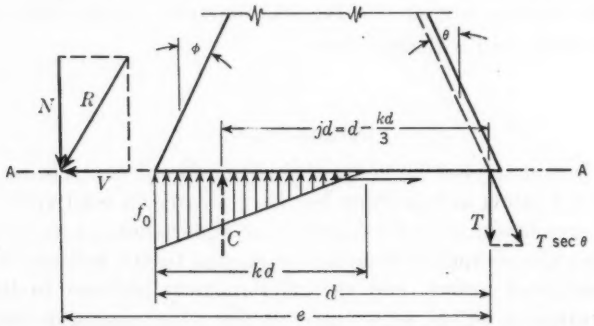


Fig. 1.—Forces and Stresses Acting on Part of a Wedge-Shaped Member

indicated by C, and the component, normal to the section A–A, of the total tension in the reinforcement by T. The compressive unit stress normal to and at the edge of the section is shown as f_0.

In Fig. 2 assume that plane A–A′ and B–B′ bound a part of a wedge-shaped reinforced concrete member. The dashed line on the side of A′B′ represents the reinforcement. Assume that plane B–B′ is held in position while plane A–A′ is rotated through a very small angle α. The deformation, or change in length, of the reinforcement for length $\overline{rt}$ is $\overline{tu}$. Using the law of sines one may write

$$\frac{\overline{tu}}{\sin (90 - \alpha)} = \frac{d\,(1 - k) \tan \alpha}{\sin (90 + \alpha - \theta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

[4] ASA—Z10a—1932.
[5] ASA—Z10.3—1942.

or

$$\overline{tu} = \frac{d\,(1-k)\tan\alpha\cos\alpha}{\cos(\theta-\alpha)} \approx \frac{d\,(1-k)\,\alpha}{\cos\theta} \quad \ldots\ldots\ldots\ldots (2)$$

since α is so small that $\tan\alpha = \sin\alpha = \alpha$ and $\cos\alpha = 1$, approximately. In Eqs. 1 and 2: d = the distance from the compression surface of the member to

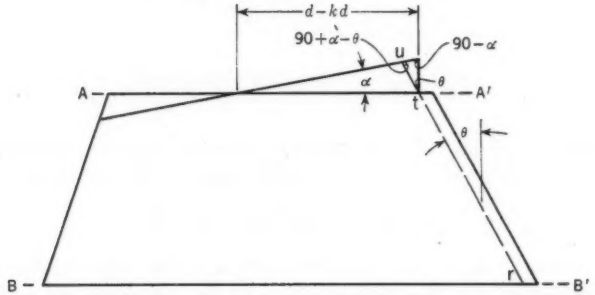


FIG. 2.—RELATION BETWEEN ANGLE CHANGE DUE TO BENDING AND DEFORMATION IN STEEL

the center of the tensile reinforcement measured in the plane of section A–A; and k = the ratio of the distance from the neutral axis to the outer compressive fiber of a reinforced concrete member to the distance, d.

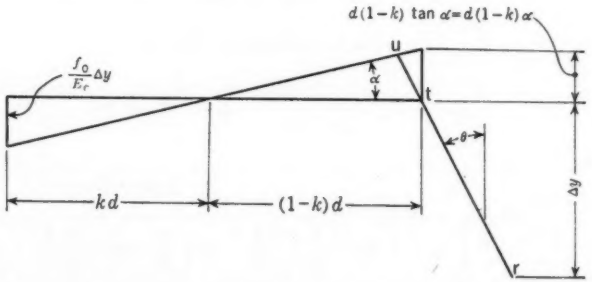


FIG. 3.—RELATION BETWEEN DEFORMATION OF CONCRETE AND DEFORMATION OF STEEL

Let Fig. 3 represent the deformations in length Δy. The angle α may be expressed in terms of the compressive unit stress in the concrete, f_0, normal to section A–A, as

$$\alpha = \frac{f_0\,\Delta y}{k\,E_c\,d} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and, multiplying both sides of Eq. 3 by $d\,(1-k)$, Eq. 3 becomes

$$d\,(1-k)\,\alpha = \frac{f_0}{E_c}\,\frac{1-k}{k}\,\Delta y \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now

$$\overline{tu} \approx d\,(1-k)\,\alpha\sec\theta = \frac{f_0}{E_c}\,\frac{1-k}{k}\sec\theta\,\Delta y \quad \ldots\ldots\ldots\ldots (5)$$

and

$$\overline{rt} = \sec\theta\,\Delta y \qquad (6)$$

Therefore, the unit strain in the reinforcement can be expressed as

$$\frac{\overline{tu}}{\overline{rt}} = \frac{f_0}{E_c}\frac{1-k}{k} \qquad (7)$$

and the unit stress in the reinforcement as

$$f_s = E_s\frac{f_0}{E_c}\frac{1-k}{k} = n f_0\frac{1-k}{k} \qquad (8)$$

From a consideration of equilibrium (namely, the summation of vertical forces is equal to zero), the following expression may be written:

$$\tfrac{1}{2} f_0\, b\, k\, d = f_s\, p\, b\, d\cos\theta + N \qquad (9)$$

in which b = the width of the member; and p = the ratio of the area of the tensile reinforcement to the effective area of the concrete, $b\,d$. The area of the reinforcement is taken on a plane normal to the axis of the reinforcement.

Upon taking moments, about the point of application of the normal component of the resultant tensile stress in the reinforcement, of the forces acting on the section it is found that

$$\tfrac{1}{2} f_0\, k\, j\, b\, d^2 = N\, e \qquad (10)$$

Solving for N in Eq. 10 and substituting in Eq. 9,

$$\frac{1}{2} f_0\, k\, b\, d\left(1 - j\frac{d}{e}\right) = f_s\, p\, b\, d\cos\theta \qquad (11)$$

or

$$\frac{f_0}{f_s} = \frac{2\,p\cos\theta}{k\left[1-\left(1-\frac{k}{3}\right)\frac{d}{e}\right]} = \frac{2\,p\frac{e}{d}\cos\theta}{k\left(\frac{e}{d}-1+\frac{k}{3}\right)} \qquad (12)$$

Using Eq. 8 and Eq. 12 and eliminating $\frac{f_0}{f_s}$, one can write

$$k^2\left(\frac{e}{d}-1+\frac{k}{3}\right) = \frac{e}{d}(1-k)\,2\,p\,n\cos\theta \qquad (13)$$

or

$$\frac{k^3}{3} + k^2\left(\frac{e}{d}-1\right) + k\left(2\,p\,n\frac{e}{d}\cos\theta\right) = 2\,p\,n\frac{e}{d}\cos\theta \qquad (14)$$

Eq. 14 can be rearranged and a solution for $\frac{d}{e}$ computed as follows:

$$\frac{d}{e} = \frac{k^2 + k\,2\,p\,n\cos\theta - 2\,p\,n\cos\theta}{k^2\left(1-\frac{k}{3}\right)} = \frac{3}{3-k} + \frac{6\,(k-1)}{k^2\,(3-k)}\,p\,n\cos\theta \qquad (15)$$

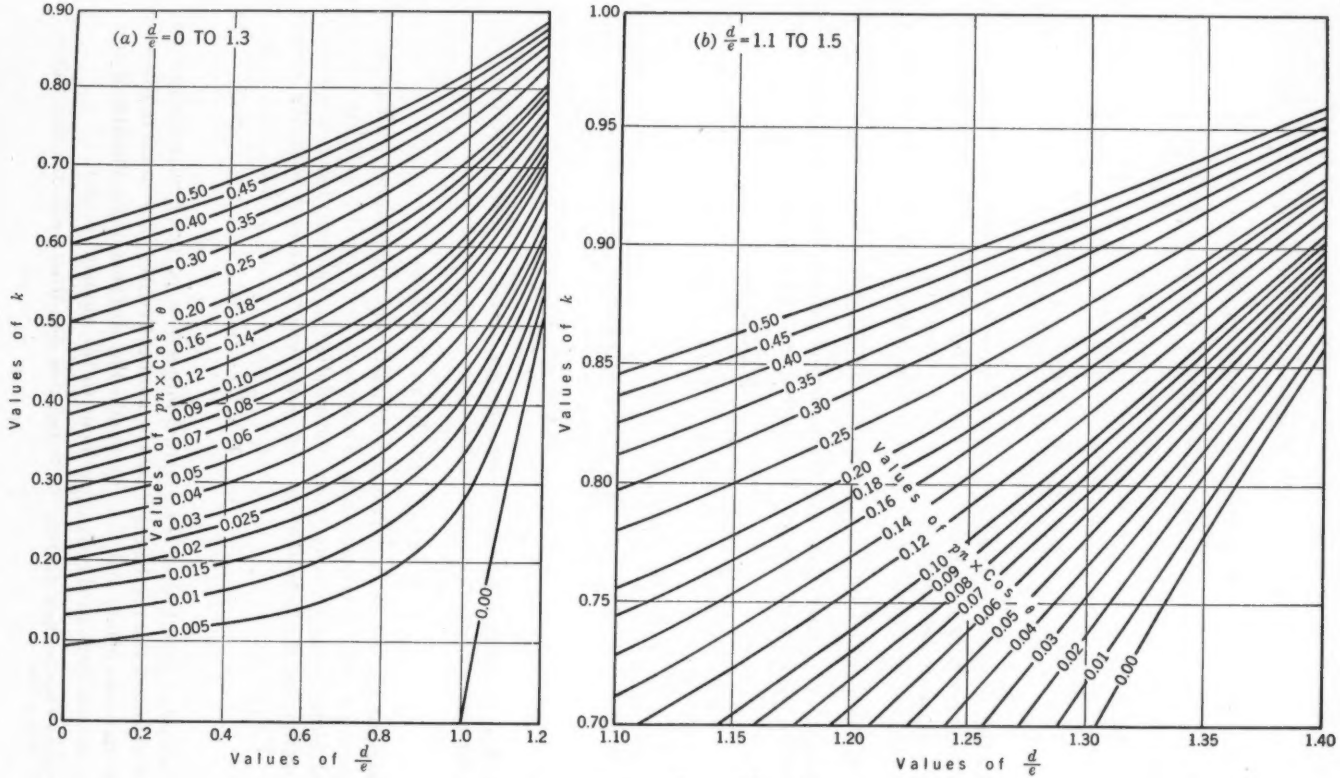


FIG. 4.—CURVES OF $\frac{d}{e} = \frac{3}{3-k} + \frac{6(k-1)}{k^2(3-k)} p\, n \cos\theta$

Eq. 15 has been used to plot the curves for various ($p\ n \cos\theta$)-values shown in Fig. 4. The $\frac{d}{e}$-values are the abscissas and the k-values the ordinates. The value of j may be computed by using the expression $j = 1 - \frac{k}{3}$.

The following expressions are presented for use in computing the maximum normal stresses:

$$f_c = f_0 \sec^2\phi = \frac{2\,N\,e\sec^2\phi}{k\,j\,b\,d^2} \qquad (16a)$$

and

$$f_s = n f_0 \frac{1-k}{k} = \frac{2\,n\,N\,e\,(1-k)}{k^2\,j\,b\,d^2} \qquad (16b)$$

In Eq. 16*a* it will be noted that the maximum compressive unit stress in the concrete is given as $f_c = f_0 \sec^2\phi$. This expression[6] is for the unit stress on a plane normal to the compression face of the member.

In the case of a tee-shaped section in which the flange is in compression, $j\,d$ may be taken equal to $d - \frac{t}{2}$, approximately, in which t is the thickness of the flange. One may then use the following approximate expressions when computing the unit stresses due to direct stress and bending:

$$f_c = f_0 \sec^2\phi = \frac{2\,N\,e\sec^2\phi}{t\left(d - \frac{t}{2}\right)b_1} \qquad (17a)$$

and

$$f_s = \frac{N\left(e - d + \frac{t}{2}\right)\sec\theta}{A_s\left(d - \frac{t}{2}\right)} \qquad (17b)$$

in which b_1 = width of the compressive flange; and A_s = area of effective tensile reinforcement, for this case in width b_1.

Shear and Bond Stresses

Imagine Fig. 5 to be a part of a reinforced concrete member with a variable cross-section area. Assume that two sections are passed parallel to the horizontal plane a distance dy from one another. Taking moments about the intersection of the resultant of the compressive stresses with the section, $R\,r = T\,z'$; or

$$T = \frac{R\,r}{z'} \qquad (18)$$

in which: R = resultant of the external forces; r = perpendicular distance from line of action of the resultant, R, to the moment center; T = total tensile stress in the reinforcement; and z' = perpendicular distance from the line of action of the tensile force in the reinforcement to the moment center.

[6] "Earth Pressure, Walls and Bins," by William Cain, John Wiley & Sons, Inc., New York, N. Y. 1st Ed., 1916, p. 240.

Considering changes to take place only in T, r, and z', the change in tensile stress in the reinforcement in distance dy may be expressed as

$$dT = R\,\frac{z'\,dr - r\,dz'}{(z')^2} = \frac{R\,dr}{z'} - \frac{R\,r\,dz'}{(z')^2} \quad \ldots\ldots\ldots\ldots (19)$$

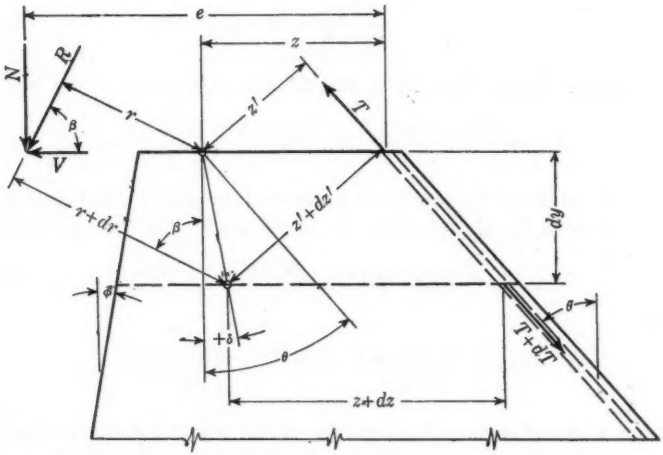


FIG. 5.—PART OF WEDGE-SHAPED MEMBER SHOWING MOMENT CENTERS

The following expressions for dr, dz', and z' may be derived from the geometrical relations (see Fig. 5):

$$dr = dy \cos\beta + dy \tan\delta \sin\beta \quad \ldots\ldots\ldots\ldots (20a)$$

$$dz' = dy \sin\theta - dy \tan\delta \cos\theta \quad \ldots\ldots\ldots\ldots (20b)$$

and

$$z' = z \cos\theta \quad \ldots\ldots\ldots\ldots (20c)$$

Substituting Eqs. 20 in Eq. 19, the expression for dT is

$$dT = \frac{R\,dy}{z\cos\theta}(\cos\beta + \tan\delta \sin\beta) - \frac{R\,r\,dy}{z^2\cos^2\theta}(\sin\theta - \tan\delta\cos\theta) \ldots (21)$$

The differential tension, dT, in the reinforcement produces a unit shearing stress, v, in the adjacent concrete, which is parallel to the reinforcement, and an equal unit shearing stress v at right angles to the reinforcement.

If b is the width of the member, one may write

$$b\,v\,\frac{dy}{\cos\theta} = dT \quad \ldots\ldots\ldots\ldots (22)$$

Substituting Eq. 21 in Eq. 22 and simplifying:

$$b\,v = \frac{R}{z}(\cos\beta + \sin\beta\tan\delta) - \frac{R\,r}{z^2}(\tan\theta - \tan\delta) \ldots\ldots (23)$$

From Fig. 4(*b*), also noting Eq. 18, it may be seen that

$$R \cos \beta = V \quad \ldots\ldots (24a)$$

$$R \sin \beta = N \quad \ldots\ldots (24b)$$

and

$$\frac{R\,r}{z \cos \theta} = T \quad \ldots\ldots (24c)$$

Upon substitution of Eqs. 24 in Eq. 23, the expression for $b\,v$ becomes

$$b\,v = \frac{V + N \tan \delta}{z} - \frac{T\,(\tan \theta - \tan \delta)}{z} \cos \theta \quad \ldots\ldots (25)$$

or, if the moment $R\,r$ of the external forces with respect to the selected moment center be indicated by M_0 (that is, $M_0 = z\,T \cos \theta$), Eq. 25 may be written as

$$b\,v = \frac{V + N \tan \delta}{z} - \frac{M_0\,(\tan \theta - \tan \delta)}{z^2} \quad \ldots\ldots (26)$$

or

$$v = \frac{V + N \tan \delta}{b\,j\,d} - \frac{M_0\,(\tan \theta - \tan \delta)}{b\,(j\,d)^2} \quad \ldots\ldots (27)$$

in which $j\,d$ = the distance, in the plane of the section, between the resultant of the compressive stresses and the resultant of the tensile stresses.

If it is desired to use the moment about the reinforcement as was done previously, the expression for M_0 becomes

$$M_0 = z\,T \cos \theta = R\,r = N\,(e - z) = N\,(e - j\,d) \quad \ldots\ldots (28)$$

and substitution in Eq. 27 results in

$$v = \frac{V}{b\,j\,d} + \frac{N \tan \theta}{b\,j\,d} - \frac{N\,e}{b\,(j\,d)^2}\,(\tan \theta - \tan \delta) \quad \ldots\ldots (29a)$$

In the case of a tee-shaped cross section, if the tee is on the compressive side, the position of the resultant of the compressive stresses changes so slightly with respect to the tee that $\tan \delta$ may be taken equal to $\tan (-\phi)$ without appreciable error. Eq. 29*a* can then be written as

$$v = \frac{V}{b'\,j\,d} + \frac{N \tan \theta}{b'\,j\,d} - \frac{N\,e}{b'\,(j\,d)^2}\,(\tan \theta + \tan \phi) \quad \ldots\ldots (29b)$$

in which b' = the width of the stem of the tee-shaped member.

The expression for the unit bond stress, u, may be written

$$u = \frac{dT}{\frac{dy}{\cos \theta}\,\Sigma_o} = \frac{b\,v}{\Sigma_o} \quad \ldots\ldots (30)$$

since, from Eq. 22, one has $\dfrac{dT}{\dfrac{dy}{\cos \theta}} = b\,v$. For rectangular sections, using

Eq. 29*a*, the expression for u becomes

$$u = \frac{V}{j\,\Sigma_o\,d} + \frac{N\tan\theta}{j\,\Sigma_o\,d} - \frac{N\,e}{(j\,d)^2\,\Sigma_o}(\tan\theta - \tan\delta) \ldots\ldots\ldots (31)$$

Eq. 31 may be used for tee-sections by substituting ϕ for δ in the last quantity.

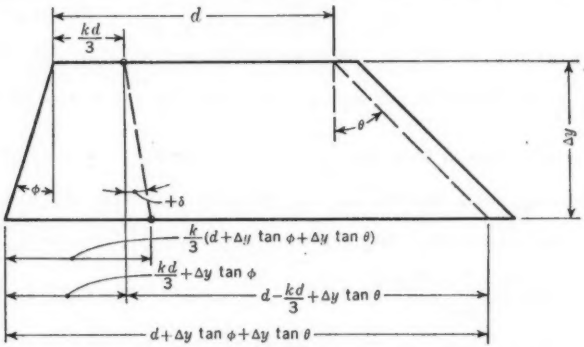


FIG. 6.—CHANGE IN POSITION OF RESULTANT COMPRESSIVE STRESSES

An approximate expression for $\tan\delta$ may be derived in the following manner: Since k varies so slightly with y, especially over a distance Δy, one may write (noting Fig. 6),

$$\Delta y\tan\delta = \frac{k}{3}[d + \Delta y\,(\tan\theta + \tan\phi)] - \frac{k\,d}{3} - \Delta y\tan\phi \ldots\ldots (32)$$

Dividing by Δy the expression for $\tan\delta$ becomes

$$\tan\delta = \frac{k}{3}(\tan\theta + \tan\phi) - \tan\phi = k\left(\frac{\tan\theta}{3}\right) - j\tan\phi \ldots\ldots (33)$$

ILLUSTRATIVE EXAMPLE

Assume that it is desired to determine the unit stresses at section A–A in the part of the structure shown in Fig. 7. The force, R, per foot of member

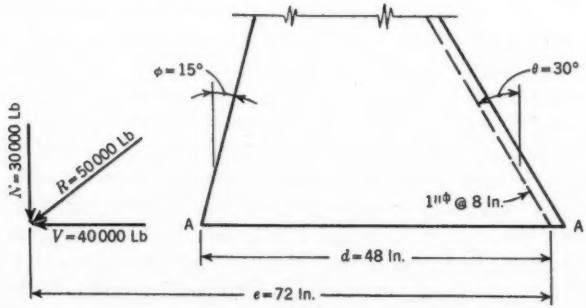


FIG. 7.—ILLUSTRATIVE EXAMPLE

normal to the plane of the paper, is resolved into vertical (N) and horizontal (V) components. Other data are given in Fig. 7. The computation procedure

is as follows: A_s per foot = 1.5 sq in.; Σ_o per foot = 6 in.; $p = \frac{1.5}{48 \times 12} = 0.0026$; $n = 10$ (given); and $p\,n = 0.026$. Enter Fig. 4(*a*) with $p\,n \cos\theta = 0.0225$ and $\frac{d}{e} = 0.667$; read $k = 0.29$; and compute $j = 1 - \frac{k}{3} = 0.903$. Use Eq. 10 to compute $f_o = \frac{2 \times 30{,}000 \times 72}{0.29 \times 0.903 \times 12 \times 48^2} = 596$ lb per sq in.

From Eq. 16*a*, $f_c = 596 \times 1.072 = 639$ lb per sq in.; and, from Eq. 8, $f_s = 6{,}390 \frac{0.71}{0.29} = 15{,}630$ lb per sq in. A value for $\tan\delta$ can be determined by use of Eq. 33 as equal to $0.29 \frac{0.577}{3} - 0.903 \times 0.268 = -0.186$.

The shearing unit stress (as a measure of the diagonal tension) can be computed using Eq. 29*a*; thus, $v = \frac{40{,}000 + 30{,}000 \times 0.577}{12 \times 0.903 \times 48} - \frac{30{,}000 \times 72 \times 0.577 + 0.186}{12\,(0.903 \times 48)^2} = 37$ lb per sq in. The bond unit stress can be computed by use of Eq. 30 as $u = \frac{12 \times 37}{6} = 74$ lb per sq in.

Conclusions

Formulas have been developed for compressive bond and shearing stresses in the concrete and for tensile stress in the reinforcement for wedge-shaped reinforced concrete members. These expressions are applicable to members subject to bending and direct stress. The formulas are correct in so far as the assumptions used in the ordinary flexural theory are applicable to reinforced concrete.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

REPORTS

PRINCIPLES OF DEPRECIATION

REPORT OF THE SPECIAL COMMITTEE AUTHORIZED BY THE BOARD OF DIRECTION TO ANALYZE AND DISCUSS THE 1943 REPORT OF THE NATIONAL ASSOCIATION OF RAILROAD AND UTILITIES COMMISSIONERS' COMMITTEE ON DEPRECIATION

FOREWORD

For many years the Society has supported a busy schedule of studies to improve and crystallize the best practice in the application of depreciation theories as related to civil engineering projects. In 1911 a Special Committee "To Formulate Principles and Methods for the Valuation of Railroad Property and Other Public Utilities" was appointed. It presented a Progress Report in 1914, and in 1916 brought out an extensive Final Report. This was widely discussed and thus was developed into final form as it appears in 310 pages of the 1917 *Transactions*. This treatment was widely acclaimed as setting up fundamental engineering principles applied to such matters as maintenance and operating costs, amortization, depreciation, obsolescence, valuation, and similar elements. The Society's interest has since continued, as expressed through a number of committee and Division activities.

Recently there appeared a 277-page printed report by a Committee of the National Association of Railroad and Utilities Commissioners (NARUC) which deviated from some of the long-standing principles previously enunciated in Society publications and in general use. Having knowledge of this report, the Board of Direction requested that opportunity to present the contrary views of civil engineers be afforded. President Pirnie then appointed a Special Committee with instructions to study the report objectively and to present its findings to the Board of Direction of the Society. In order that this committee might submit its findings, the NARUC committee granted an extension of time to April 30 before taking further official action.

The tentative report of the Society's Special Committee was submitted to the Board at its St. Louis Meeting, April 24. After considering the findings,

NOTE.—Please forward all comments on this report to Society headquarters, 33 West 39th Street, New York, N. Y., for transmission to the Committee.

the Board instructed that this document be printed for the benefit of all members. It is presented herewith. A summary quoted from pages XIV to XIX is appended.

A still further development is that the subject of valuation, depreciation, and rate making is to be made a special item of technical consideration at the time of the Annual Convention of the Society in Cleveland, Ohio, on July 21, 1944. In view of the active diversity of views occasioned by the NARUC report, it is especially urged that all who can do so submit written comments for the study of the Special Committee prior to that meeting. Such comments, forwarded to headquarters, will be promptly transmitted to the committee.

Report of the Special Committee

A Special Committee of the Society was appointed in February, 1944, for the purpose of analyzing and commenting upon the report of the Committee on Depreciation of the National Association of Railroad and Utilities Commissioners (NARUC) issued in 1943. The Society's Special Committee has studied that report and respectfully submits its analysis and recommendations.

The NARUC report embraces some 277 pages. The Special Committee of the Society has deemed it best to limit its report to a discussion of only the most fundamental differences which it has with the report of the NARUC committee. The principal divergencies of views are as follows:

1. The Society's committee takes exception to the fundamental concept of depreciation as set forth in Conclusion 2 of the summary of the NARUC committee report (Appendix).

2. The NARUC report states (Conclusion 26, Appendix):

> "The straight-line method is generally recommended for public utility accounting and financial purposes and also for the computation of both depreciation expenses and accrued depreciation for purposes of rate making."

and again (Conclusion 21, Appendix),

> "The straight-line method is in very general use and is in the ascendancy in the public utility field."

(Conclusion 22, Appendix),

> "The straight-line method is simpler than, and is not as seriously affected by errors in estimates of service lives as the interest methods."

3. The NARUC report repeatedly states that actual depreciation conforms to the depreciation reserve accrued by the straight-line method. The following excerpts are illustrative (page 169, NARUC report):

> "The depreciation reserve shows the results to date of the effective depreciation factors,"

(on page 170, NARUC report),

> "Where cost is the gross base, then the reserve should be deducted from it in arriving at the rate base, for the reserve alone measures actual depreciation."

and (Conclusion 39, Appendix),

> "A properly computed depreciation reserve is the best measure of accrued or existing depreciation, since such reserve reflects that part of the cost of the property in service which relates to the exhausted or expired economic or service life."

4. The NARUC report states (Conclusion 24, Appendix):

> "* * * The total dollar charges to customers in the long run, however, will be greater under the sinking-fund or compound-interest methods than under the straight-line method."

5. At numerous places in the NARUC report reference is made to the necessity of correcting depreciation reserves of utilities when such reserves differ from those resulting from the application of the straight-line method, and the recommendation is that the deficiency be made up through withdrawals from surplus or a special section of the income account, or some other source. This is illustrated by the following quotation (Conclusion 30 in the Appendix):

> "The depreciation reserve should not be readjusted gradually through modification of the annual depreciation rates when the difference between the book reserve and the proper reserve is substantial. When the difference is not substantial it is satisfactory to spread the remaining net cost of the properties over their remaining lives."

The foregoing examples and other features of the NARUC report are discussed more fully subsequently.

Definition of Depreciation

The Society's committee believes that of the many definitions of depreciation that of Mr. Chief Justice Hughes in the Lindheimer case (292 U. S. 151), as follows:

> "Broadly speaking, depreciation is the loss, not restored by current maintenance, which is due to all the factors causing the ultimate retirement of the property. These factors embrace wear and tear, decay, inadequacy and obsolescence. Annual depreciation is the loss that takes place in a year."

and that of the existing NARUC uniform System of Accounts reading:

> "'Depreciation' as applied to depreciable utility plant, means loss in service value not restored by current maintenance incurred in connection with the consumption or prospective retirement of utility plant in the course of service from causes which are known to be in current operation and against which the utility is not protected by insurance. Among the causes to be given consideration are wear and tear, decay, action of the elements, inadequacy, obsolescence, changes in the art, changes in demand and requirements of public authorities * * *."

—are the best.

As contrasted with these definitions, which reflect the latest authoritative statement of the U. S. Supreme Court, the 1943 NARUC depreciation committee report describes depreciation in Conclusion 2 (Appendix) as follows:

> "The significance of the depreciation phenomenon may be described as follows:
>
> "(a) Depreciation is the expiration or consumption, in whole or in part, of the service life, capacity, or utility of property resulting from the action of one or more of the forces operating to bring about the retirement of such property from service;
>
> "(b) The forces so operating include wear and tear, decay, action of the elements, inadequacy, obsolescence, and public requirements;
>
> "(c) Depreciation results in a cost of service."

and again in Conclusion 9 (Appendix),

> "Depreciation is the using up of the economic or service life of an asset regardless of the specific factor which may ultimately cause the asset to be retired. It is the exhaustion of service life, not the particular cause of retirement, that is important."

Attention is directed to the foregoing definitions and to the fact that the definitions heretofore in effect and approved by the U. S. Supreme Court refer to "loss in service value not restored by current maintenance," which in the new committee definition is referred to as the "consumption in whole or in part of the service life capacity or utility of property." This is a change which in effect predicates the deduction from cost new upon "consumption of service life capacity or utility" rather than upon loss in service value not restored by current maintenance.

The Society's committee cannot subscribe to the implied opinion of the NARUC committee that remaining service life and remaining service value are synonymous, although there usually is a relationship (not direct, however) between the two.

There are other definitions and terms used in the NARUC report to which the Society's committee takes exception, but to which it would seem unnecessary to devote time and space in this discussion.

It would appear to serve no useful purpose in the discussion of the NARUC report to refer specifically to many minor points of divergence with the theories outlined in the NARUC report or their application or the mathematical examples with which the Society's committee is not in accord, or the unrealistic conclusions to which the application of the principles of the NARUC report would lead in many instances.

The Concept of Depreciation

The NARUC report discusses the several aspects of depreciation and depreciation accounting at length. In the opinion of the Society's committee it fails to present adequately the fundamental conception that loss of value to property through age and use is inescapable and that the extent of that loss of value at any date should be the basis of all accounting and financial policies with reference to depreciation.

In the last analysis the estimate of depreciation for any period must rest on human judgment. The most reliable estimate of depreciation is one made by qualified persons who have examined the particular property, studied the mortality characteristics of the kinds of units of which it is composed, and considered the prospects of future use of each of the units (or groups of like units when there are included in the property large numbers of these).

Moreover, the property ought to be re-examined and the depreciation redetermined at appropriate intervals in order to take into account any changes in the probable future usefulness of any of the units.

Accrued depreciation existing in any particular property at any particular time is a matter of fact and not of theory. True it cannot be determined with mathematical accuracy by formula, but it is subject to determination within reasonably close limits by men experienced in the design, construction, and operation of utility properties. It is basically an engineering determination and the Society's committee contends that it is absolutely impossible to arrive at any reasonable answer by any other process, and certainly not by any theoretical calculation.

The NARUC committee report goes to considerable length to discredit what it refers to as "observed depreciation," and which it explains as that depreciation which can be seen by the naked eye. No one competent to determine depreciation contends that depreciation can be seen with the naked eye. The modern methods of determining depreciation include some observations of the property, but they include much more. This modern method might properly be called the engineering method of estimating actual depreciation.

The engineering determination of accrued depreciation involves many operations and analyses in addition to what has been referred to as "observing" it. Operating records are studied, including past and probable future growth of the system, the types of equipment, their functional features such as adequacy and obsoleteness, and many other elements that would affect probable future usefulness. Poles are inspected and butts measured to determine their actual condition; maintenance records are examined to determine the care that has been given to the upkeep of the property; deferred maintenance, if any, is ascertained; records of past retirements are analyzed, and in fact all data having a bearing on the future usefulness of the units involved in the particular property in question are studied.

When the depreciation of a property has been determined in the aforedescribed manner, the actual depreciation thus established is factual and affords a sound basis for appropriate financial and accounting procedures. The Society's committee contends that accrued depreciation can only be ascertained accurately by this method.

Straight-Line Depreciation

The NARUC report repeatedly expresses preference for the "straight-line" method of depreciation accounting, ostensibly on the grounds of simplicity. The Society's committee emphasizes that actual depreciation is factual and that accounting, to be correct, must reflect the facts of actual depreciation. The

depreciation is in the property—not in the accounts. Therefore, it is inadvisable that any method of accounting for actual depreciation should be adopted for no other reason than that it is simple.

No exception is taken to the use of age-life methods as a help in estimating the annual charges for depreciation where sufficient retirement data are available from any particular utility system to enable past useful lives of retired property to be determined with a fair degree of accuracy. Accurate retirement data that could be considered as mass data cover a relatively small number of the numerous units involved in public utility property. Also, the data that are available, generally, cover items that are different and usually inferior to those presently existing in the usual utility property.

As an example, most retirement data on poles do not reflect the service that can be gotten from a pole having modern preservative treatment. Retirement data on water mains of necessity cover pipe installed and used during a period when modern coatings were not available, and modern practice in water treatment to correct for corrosion and incrustation was not yet developed. Electric meters that have been retired for other than functional reasons were old types developed in the early years of the electric industry. The same situation exists for gas and water meters, and most of the equipment and material in all utilities. Included in the available retirement data are some modern items that have been retired due to casualty or functional reasons, but such data certainly cannot be considered as more than very roughly indicative of what may be expected in the future.

Estimates of future useful lives made with the help of pertinent retirement data may be considered indicative where the data are sufficiently comprehensive. The available data may be meager or nonexistent on many of the items involved, and exactly similar units will have varying life histories in different properties. As a large percentage of retirements are made for functional reasons, such as inadequacy, obsolescence, action of public authorities, etc., it is manifest that factors such as the character of the community served, its rate of growth, the cost of fuel, labor and material, etc., have definite bearings on usefulness of items involved in utilities serving them. An operating unit that, due to size or to operating costs, would be retired in one locality, may have several times as many years of useful service in some other system.

Incidentally, the Society's committee challenges the contention of simplicity because the data employed in computations in connection with actual depreciation are commonly taken from tables; and, in any case, inability to make calculations of simple or compound interest should not be sufficient grounds for using a depreciation accounting method that distorts the facts for certain classes of property.

That method of accounting for depreciation for any particular unit of property is best which most closely harmonizes with the factual accrued depreciation of the unit. In general, depreciation emerges from the twilight zone of maintenance for very short lived units, which are usually financed directly through operating expenses, to the very small annual requirements of units of property having long lives. With the short lived types (10 years or less) it is

not so important what annual depreciation method is used, but with the long lived units the importance greatly increases.

Straight-line depreciation does not conform to the facts for long life property whose accrued depreciation experience more nearly parallels the accumulation of a sinking fund.

Depreciation reserves are accrued to meet a future obligation and where that period is long the use of the straight-line method is unsound economically. With long life property, such as dams, main line railroad embankments, cast-iron water works distribution systems, masonry structures, etc., it is manifestly sound business to provide for the obligation maturing at that remote time by an annuity rather than by a uniform allowance on a straight-line basis. The actual depreciation which the long lived unit has suffered will usually be much less than the accumulation in a straight-line reserve.

Straight-Line Depreciation not Cheapest for Customers

The NARUC report purports to show that the "straight-line" method of depreciation accounting will require the lowest contribution from the customers of any of the methods considered. This is manifestly incorrect. The customers' contribution is entirely independent of the method of accounting followed. Moreover, all methods of depreciation accounting will result in identical contributions from the customers if applied correctly with due regard to the time value of money. Therefore the statement in Conclusion 24 (Appendix) that the

> "* * * total dollar charges to customers in the long run * * * will be greater under the sinking-fund or compound-interest methods than under the straight-line method"

is incorrect and misleading and has no basis other than the assumption that the rate payer would be less provident in getting interest on his money than the utility will be in investing the sinking fund. If the depreciation accounting is correct, the total charge against operation for any unit of property is its value new less net salvage value.

Depreciation Reserve

The depreciation reserve is created by credits from annual charges against operations for depreciation. Retirements are charged to the reserve, and the balance in it at any one time should at least equal the amount of accrued depreciation existing at the same time. The reserve should normally exceed the accrued depreciation by a reasonable margin, the amount depending upon circumstances existing in the particular utility. It should be large enough to absorb casualties (not covered by insurance) which are properly not included in the determination of accrued depreciation.

The things or circumstances that bring about depreciation and retirement of property units, as stated previously in this report of the Society's committee, vary in each property and from year to year in any one property. It is entirely feasible to have the amount of accrued depreciation determined at regular

intervals varying from 5 to 10 years, depending upon the property involved, and if it is found that the balance in the depreciation reserve is either too high or too low, adjustment can be made in the annual depreciation allowances to return the two into proper relationship. Due to the nature of the reserve (including, as it should, casualties and other elements that have no place in measurement of accrued depreciation) there can be no justification for deducting it as accrued depreciation.

It is submitted that the actual depreciation should control the size of the reserve, rather than that the reserve should determine the actual accrued depreciation.

The depreciation reserve is created to take care of retirements, and it seems proper that property retired should be retired at the cost recorded in the books and records of the property. The reserve is created from charges to operations, as depreciation is undoubtedly an incremental part of the cost of service. The Society's committee believes that the reserve is a definite liability and is in a sense a trust fund, held until such time as replacements are necessary, and that in the case of a continuing utility it should not be considered a return of capital to be disposed of as desired.

Uniformity of treatment and administrative convenience in dealing with depreciation would be desirable if equitable; but neither uniformity nor administrative convenience should control over the facts of a particular situation. The NARUC report under discussion would make abstract theory, rigid uniformity, and administrative convenience controlling. The Society's committee believes that the results of such action would be ruinous to the public utility business. As already stated, actual depreciation existing in physical property of a public utility is a fact. It is the kind of fact that cannot be determined merely from a set of books or tables, or by any other means than study on the ground by qualified, trained engineers or operators with engineering training who follow approved methods.

Abuses of the engineering determination of accrued depreciation are no more probable than manipulation of the basic figures upon which theoretical age-life determinations are made. The possibilities of improper use of any method should not be used as justification for abandoning it if it produces the correct answer.

Retroactive Adjustment of the Depreciation Reserve

The report of the NARUC Committee on Depreciation recommends the setting up of depreciation reserves on the basis of straight-line requirements. It further recommends that where the reserve computed by the straight-line method substantially exceeds the reserve currently in the property that the adjustment necessary to bring the current reserve up to the depreciation reserve computed on the straight-line basis should be through charge to surplus or special section of the income account.

The NARUC report recognizes that this adjustment presents many practical difficulties. The Society's committee feels that in view of the fact that practically all utilities have been subject to governmental regulation over a period of

two to four decades and the reserves currently carried on the books of the utilities have been accumulated, credited and charged in compliance with the regulation of such bodies, the adjustment of the current reserve upward so as to equal those computed on the straight-line basis by withdrawal from surplus or otherwise is not only generally financially impracticable but is also ethically unjust.

Present reserves should, and usually do, reflect rates charged and collected from consumers. To arbitrarily increase the reserves at this late date by withdrawal from surplus is to require the utility rather than its consumers to make good the cost of property used up in their service. This procedure is in conflict with the basic principle of depreciation as well as plain justice.

The reserve requirements should be measured by the actual needs of the utility and not by an arbitrary straight-line determination which in the case of many of the best utilities in the United States would require reserves substantially double those which a long background of experience under regulation has determined as adequate.

It is the opinion of the Society's committee that Conclusion 40 (Appendix) is ambiguous, is being variously interpreted, and is impracticable of equitable application. It involves consideration of many matters other than depreciation.

General

In this report the Society's committee has endeavored to cover only a few of the most important features of the NARUC committee report. That report contains many valuable data, and can be considered a definite contribution to the subject of depreciation. The Society's committee has given the report careful study and has reached the conclusion that desirable as it may be, from the standpoint of regulatory simplicity, it is impossible to determine depreciation, either annual or accrued, by means of theoretical actuarial calculations because of the many variables existing in the elements causing depreciation. It would be fine if a crystal ball could be used, and the future accurately determined, but this has not been possible to date and, as far as can be foreseen, never will be.

Depreciation, its causes and effects, are pretty generally understood and should be accepted as a phenomenon differing in every property, and not subject to mathematical standardization.

Respectfully submitted,
T. R. Agg
N. T. Veatch, Jr.
L. R. Howson
Henry E. Riggs, *Chairman*

Special Depreciation Committee of the Board of Direction

April 15, 1944

APPENDIX

Summary of the Report of the Committee on Depreciation, National Association of Railroad and Utilities Commissioners

(Pages xiv to xix of the 1943 NARUC Report)

The more important conclusions and recommendations contained in this report are summarized briefly below to afford a convenient general preview of the substance of the report as a whole. In summary statements it is obviously not possible to capture all the refinements of meaning or all the particular qualifications set forth in the main text. Therefore these statements are to be taken as condensed expressions of the ideas and principles that are developed more fully in the report.

A. Nature of Depreciation

1. The modern view of depreciation emphasizes the concept that the purchase of capital goods is in essence a purchase of future services.
2. The significance of the depreciation phenomenon may be described as follows:
   (a) Depreciation is the expiration or consumption, in whole or in part, of the service life, capacity, or utility of property resulting from the action of one or more of the forces operating to bring about the retirement of such property from service;
   (b) The forces so operating include wear and tear, decay, action of the elements, inadequacy, obsolescence, and public requirements;
   (c) Depreciation results in a cost of service.
3. Depletion is the using up of wasting assets—natural resources, such as timber, oil, natural gas, etc. Thus it is closely related to depreciation taken on a unit of production basis.
4. Amortization as applied to fixed assets is the process of writing off the cost of intangibles, such as patents, copyrights, leaseholds, etc., over a period of time. Such assets frequently have a contractual or legal life over which the cost should be spread.
5. Depreciation accounting is the process of charging the book cost of depreciable property to operations over its life.
6. The physical, functional, and accounting processes defined in the four preceding paragraphs are often referred to collectively, for convenience, as depreciation.
7. In the past several methods have been employed to account for the consumption of the service or economic life of plant assets, including the retirement, replacement, appraisal, arbitrary writedown, retirement reserve, and depreciation methods. Only the depreciation method survives in general use.
8. There are two interrelated aspects of depreciation: annual expense and accrued depreciation. Annual expense is shown in the income statement;

accrued depreciation is recorded in the depreciation reserve, a balance sheet account.

9. Depreciation is the using up of the economic or service life of an asset regardless of the specific factor which may ultimately cause the asset to be retired. It is the exhaustion of service life, not the particular cause of retirement, that is important.
10. Depreciation should not be confused with efficiency. A machine may be highly efficient when near the end of its service life and when it is almost fully depreciated in fact.
11. While the progress of depreciation may be retarded by proper maintenance, the view that maintenance prevents depreciation is unsound. Regardless of the degree of maintenance, economic life cannot be prolonged indefinitely.
12. The depreciation reserve measures that part of the cost of plant still in service which has been written off, usually as an operating expense. If the depreciation reserve has been properly determined, it measures the accrued depreciation. The depreciation reserve should preferably be shown on the asset side of the balance sheet as a deduction from depreciable plant.
13. The depreciation reserve is an account *contra* to the plant account. It is not a fund. Depreciation accounting normally results in the retention in the business of assets which otherwise might be distributed as dividends. Ordinarily these assets are not earmarked and, with minor exceptions, there is no reason why they should be.
14. The financing of replacements is not the purpose of depreciation accounting. Its purpose is to record as a cost of operations the *pro tanto* cost of property consumed therein, thus maintaining the integrity of the investment whether or not replacement occurs. Where depreciation is computed on a base higher than the actual cost of properties, accretions to capital result; where the depreciation base is lower than the actual investment, capital impairment follows.

B. Depreciable and Non-Depreciable Property

15. In practice a distinction is made between depreciable and non-depreciable assets. Non-depreciable property is, in fact, only that property which has unlimited life in service. Mere longevity or the difficulty of predicting service life should not prevent provision for depreciation according to the best information available.
16. Overheads—including interest during construction—and indirect costs, to the extent associated with depreciable plant, represent depreciable costs. Such overhead and indirect costs should be assigned to units of property so that the full costs of the property units can be spread over their respective service lives.

C. Basis of Computing Depreciation

17. Depreciation should not be based upon fair value or replacement cost. In this respect the doctrine followed in United Railways & Electric Co. v.

West (280 U. S. 234, 253, 1930) misconstrues the economic nature of depreciation and would be administratively impossible of reasonably accurate application.

18. Depreciation expense should be based upon the cost of depreciable assets. The actual cost to a utility of properties acquired as operating units or systems may differ from their original cost. Under most current systems of accounts for utilities it has been established that depreciation shall be based on original cost.
19. Any difference between the cost and original cost less depreciation is recorded in the acquisition adjustment account. Since the proper disposition of amounts in this account must depend on the circumstances which gave rise to them, no specific rules for their accounting treatment are recommended.

D. Depreciation Methods

20. Several methods have been used to compute public utility depreciation, but only three are generally recognized or accepted today. These are the straight-line, compound-interest ("modified sinking-fund"), and sinking-fund methods.
21. The straight-line method is in very general use and is in the ascendancy in the public utility field.
22. The straight-line method is simpler than, and is not as seriously affected by errors in estimates of service lives as the interest methods.
23. At all times prior to the end of the service life of a plant unit, the straight-line method results in a higher depreciation reserve than do the interest methods. In the case of long-lived plant and where a high interest rate is used under the interest methods, the difference between a reserve computed by the straight-line method and one computed according to the interest methods will be substantial. Conversely, the reserve computed according to the interest methods will approach the straight-line reserve as the interest rate approaches zero.
24. The straight-line method results in higher charges to consumers for depreciation and return (assuming the same rate of return) when property is young, and lower charges when property is old, than does either of the interest methods. It can be argued that under the "present-worth" theory the real cost to consumers in the long run will be the same, regardless of the depreciation method, provided no extra profit accrued to the utility because of a differential between the sinking-fund interest rate and the rate of return. The total dollar charges to customers in the long run, however, will be greater under the sinking-fund or compound-interest methods than under the straight-line method.
25. The compound-interest method, which is also called the present-worth method, and the sinking-fund method, which is a modification of the former, yield the same results in the aggregate, where the depreciation reserve requirement is deducted under the compound-interest method and where the interest rates under both methods are the same as the rate of return.

26. The straight-line method is generally recommended for public utility accounting and financial purposes and also for the computation of both depreciation expenses and accrued depreciation for purposes of rate making.
27. Depreciation rates may be applied to individual units of depreciable property or to groups of units. The group plan has many advantages and should be used wherever property can rationally be grouped.
28. The determination of depreciation rates according to any reasonable procedure involves the use of judgment in the application of methods. Statistical data drawn from property experience and applied with judgment based on knowledge of the property should be given much greater weight than general judgments based principally on observation of the property.

E. Current Charges and Adjustments of Reserves

29. Current charges for depreciation expense should be based upon the best possible estimates of the amount properly applicable to the period covered by an income statement, without modification for excessive or deficient charges in the past.
30. The depreciative reserve should not be readjusted gradually through modification of the annual depreciation rates when the difference between the book reserve and the proper reserve is substantial. When the difference is not substantial it is satisfactory to spread the remaining net cost of the properties over their remaining lives.
31. In principle any necessary correction of depreciation reserves should be made through surplus or a special section of the income account. However, the adjustment of inadequate depreciation reserves, while sound in theory, presents many practical difficulties. Where such deficiences are serious, it is desirable to make every effort to adjust them, although it is recognized that the application of a uniform rule without regard to what is equitable and feasible under the circumstances of individual cases might cause injury to security holders out of proportion to the long-range benefits. Therefore, it is concluded that the objective of correcting inadequate reserves should be approached with appropriate consideration of the practical effects of alternative courses of action.

F. Depreciation in Income Taxation

32. Because of certain special problems and provisions of the federal income tax laws and regulations, absolute consistency between the treatment of public utility depreciation for tax, corporate, and regulatory purposes may not now be feasible, but such consistency should be sought to the greatest practicable extent.
33. Inconsistencies between the treatment of public utility depreciation for income tax, corporate, and regulatory purposes have sometimes been unjustifiably great and often have been inadequately disclosed with attendant effects upon the tax liabilities of stockholders. Analysis of

substantial differences in the treatment of depreciation for these respective purposes is necessary, both for the protection of the investor and for fully effective regulation.

34. Since the depreciation reserve builds up more rapidly under the straight-line method than it does under the interest methods, the use of the straight-line method tends to reduce the investors' risk and therefore to lower the cost of capital.

G. Depreciation and the Investor

35. Failure to make adequate provision for depreciation results in the impairment of capital, unless an appropriate retention of surplus is made. Depreciation policies and practices should, therefore, be taken into account in the consideration of questions of public utility finance.

36. Although the immediate interests of equity- and debt-holders may sometimes conflict, and although the short-run and long-run interests of investors as a class may sometimes differ, adequate provision for depreciation is essential to the protection of the interest of investors, taken as a whole and from a long-range point of view.

H. Depreciation in Rate Making

37. The same factors which cause annual depreciation expense also cause accrued depreciation. Accordingly, the same principles should be applied in determining annual depreciation expense and in determining accrued depreciation.

38. So-called observed depreciation is not accrued or existing depreciation. Depreciation is not visually observable; only the physical effects of some of the forces which operate to cause retirement can be observed.

39. A properly computed depreciation reserve is the best measure of accrued or existing depreciation, since such reserve reflects that part of the cost of the property in service which relates to the exhausted or expired economic or service life.

40. In fixing public utility rates adequate depreciation expense should be allowed according to the service life basis and, in principle, the reserve requirement determined on the same basis should be deducted in determining the rate base. There are cases, however, where equity requires the deduction of the book depreciation reserve, even though it differs materially from the reserve requirement. The determination as to when this condition exists involves, among other things, consideration of the history of regulatory requirements in the jurisdiction and the experience and practices of the company. In other words, individual situations should be dealt with by the commission having jurisdiction in accordance with the facts in each case.

41. The use of the straight-line method in determining both depreciation expense and accrued depreciation is generally recommended for rate-making purposes.

42. The sinking-fund method may sometimes be used in rate making when it is impracticable to determine the accrued depreciation. When the sinking-fund method is used in rate proceedings, the interest rate employed should be the same as the rate of return which is applied to the undepreciated rate base.

I. Responsibility of Management and Public Accountants

43. Management and public accountants have a responsibility to indicate in financial statements the proper depreciation expense and the reserve requirement, according to their best information after appropriate study, in those instances where the book figures are inadequate or excessive in a material amount.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FLOOD FORMULAS BASED ON DRAINAGE BASIN CHARACTERISTICS

Discussion

BY CLARENCE S. JARVIS

CLARENCE S. JARVIS,[23] M. AM. SOC. C. E.[23a]—If the many hours required for the writer to grope through the maze of interrelationships disclosed in this paper may be taken as an index of the authors' labor in exploring, analyzing, and then formulating the wealth of material in the paper, the profession has received the harvest of months if not years of earnest effort. Recognizing the practical value of the authors' masterly approach and the comparatively unique if not new relationships presented, the writer somewhat reluctantly undertakes to examine background or foundational features, to trace the derivations, and to analyze the adopted factors throughout the practicable range, thus introducing both alternate and divergent views regarding major as well as minor issues—all intended as constructive suggestions throughout.

Translation of the quantities listed in Cols. 3 and 9 to 12 in Table 1 into terms of percentage ratings according to the Modified Myers formula[4] (Eq. 1) readily discloses the following relationships:

1. Quantities listed in Table 2, proposed as a supplement to Table 1, form geometric series with the following characteristics—

   (*a*) Percentage ratings for major peaks (Col. 10) are usually from 1.5 to 2.5 times those for minor peaks (Col. 9), averaging nearly 2.0.
   (*b*) Corresponding ratios between percentages listed for rare peaks (Col. 11) and for major peaks (Col. 10) are usually from 1.5 to 2.0, averaging nearly 1.7.
   (*c*) Between maximum peaks (Col. 12) and rare peaks (Col. 11), the ratios are usually from 3.2 to 4.2, averaging nearly 3.6.

NOTE.—This paper by H. B. Kinnison, M. Am. Soc. C. E., and B. R. Colby, Esq., was published in March, 1944, *Proceedings*.

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[23a] Received by the Secretary May 2, 1944.

[4] "Flood Flow Characteristics," by C. S. Jarvis, *Transactions*, Am. Soc. C. E., Vol. 89 (1926), pp. 985–1104.

TABLE 2.—PERCENTAGE RATINGS FOR USE IN THE MYERS FORMULA, EQ. 1, TO SUPPLEMENT TABLE 1, COLS. 3, 9, 10, 11, AND 12, RESPECTIVELY
(Maximum Observed Flood Ratings Are Underlined)

No.[a]	Years[b]	M (sq miles)	Col. 9		Col. 10		Col. 11		Col. 12	
			Known peaks[c]	Minor peaks[d]	Known peaks[c]	Major peaks[d]	Known peaks[c]	Rare peaks[d]	Known peaks[c]	Maximum peaks[d]
1	4	107	....	4.2	....	10.6	15.7	18.0	....	63.1
2	3	405	....	1.8	1.9	4.4	....	7.0	....	24.3
3	10	124	....	1.7	2.3	3.3	....	5.5	....	21.5
4	1	23.3	....	1.9	....	3.5	....	6.2	....	26.1
5	2	183	....	1.6	2.3	3.5	....	5.5	....	21.1
6	1	35.2	....	1.8	....	3.8	....	6.7	....	27.0
7	9	211	....	2.1	....	4.0	....	6.9	....	26.6
8	13	42.2	....	1.6	....	3.3	....	5.4	....	22.2
9	15	31.3	....	2.1	4.5	5.1	....	8.9	....	34.8
10	1	139	....	2.4	....	6.4	....	10.8	....	37.6
11	11	417	....	3.1	7.4	8.1	....	14.2	....	47.6
12	1	93.8	....	2.1	....	5.6	....	9.3	....	34.0
13	10	224	....	8.0	9.4	17.0	....	28.3	....	95.5
14	14	308	....	16.6	....	26.3	29.8	43.3	....	147.0
15	16	71.2	....	4.9	6.2	11.8	....	19.6	....	70.0
16	29	420	8.8	9.7	....	19.5	....	31.7	....	104.0
17	15	37.6	....	5.1	10.0	11.7	....	20.1	....	73.3
18	15	24.2	....	4.0	....	8.5	12.8	15.5	....	60.0
19	18	36.4	....	4.2	9.9	10.0	....	17.7	....	65.3
20	22	83.8	....	2.4	....	5.8	9.3	11.7	....	37.8
21	1	186	....	2.7	....	6.6	....	11.6	....	40.9
22	24	370	....	3.7	....	9.9	15.1	16.6	....	55.0
23	24	19.0	....	1.8	....	3.9	6.0	6.9	....	28.9
24	21	18.7	....	1.6	....	3.8	....	6.2	6.9	25.0
25	22	49.9	....	3.0	7.3	7.6	....	13.1	....	48.2
26	23	12.2	....	2.0	4.4	5.2	....	8.7	....	34.6
27	26	362	....	21.1	29.6	31.6	....	51.0	....	171.0
28	26	178	....	17.5	....	26.7	42.2	46.0	....	159.0
29	1	88.4	....	10.5	....	18.6	....	32.2	....	114.0
30	0	89.4	....	7.1	....	14.6	....	23.8	....	84.2
31	1	53.3	....	5.4	....	12.1	....	20.4	....	73.5
32	11	100	....	2.6	....	6.5	....	11.3	14.0	40.8
33	28	201	....	3.1	....	8.6	....	13.7	16.0	46.3
34	10	702	....	5.6	....	13.7	17.0	22.8	....	72.8
35	28	188	....	2.9	5.5	7.6	....	12.6	....	43.4
36	28	149	....	2.8	6.9	7.7	....	12.6	....	43.7
37	29	162	....	10.0	....	18.3	29.8	30.5	....	104.4
38	24	497	....	13.8	....	25.0	....	42.0	....	136.8
39	30	52.6	....	9.1	....	16.0	....	25.9	27.4	93.5
40	4	93.7	....	10.2	....	19.2	22.6	32.2	....	115.8
41	10	98.4	....	2.8	....	6.8	7.4	11.3	....	41.5
42	25	75.3	....	6.6	....	14.3	21.3	25.2	....	113.8
43	7	4.1	....	1.9	3.2	3.5	....	6.1	....	27.0
44	12	74.5	....	2.4	6.0	6.4	....	10.6	....	38.7
45	4	57.1	....	5.1	8.5	11.9	....	20.7	....	74.3
46	25	280	....	5.0	6.9	12.4	....	19.8	....	65.1
47	8	31.4	9.1	9.7	....	16.8	....	30.3	....	113.0
48	9	39.1	....	11.5	16.0	19.0	....	34.0	....	125.4

[a] See Fig. 10.
[b] Years of record used in the study of flood peaks (see Col. 2, Table 1).
[c] Maximum known peak discharges correspond to values in Col. 3, Table 1.
[d] Peak values computed from flood formulas (see Cols. 9 to 12, Table 1); percentage $= \frac{Q}{100\sqrt{M}}$, in which Q = peak discharge in cubic feet per second; M = drainage area in square miles (solved from a single setting of a slide rule per item).

(*d*) A notable degree of consistency pervades most of the series, manifested in a tendency to adhere either to average values or to above-average or below-average values throughout.

(*e*) Random selection of items for checking computations, mainly by log-log slide rule, resulted in satisfactory checks, within 3% or less, differences being partly attributable to the mechanics of the procedures and to cumulative errors.

(*f*) Considering all the factors concerned with flood frequencies and magnitudes, it seems advisable to re-examine all items in Table 1 with ratings according to the Modified Myers formula for "maximum" peaks (Col. 12) greater than 90%. Very few, if any, of these stations seem to deserve such high ratings according to the writer's interpretation of all available data pertinent to the locality involved.

2. Factors involved in Eqs. 6 have been platted in Fig. 14 on semilogarithmic paper throughout the practicable range for most rivers. To depict all the trends on a single drawing, it was necessary to convert the three factors *a*, dealing with pondage to percentage values, by multiplying by 100. Furthermore, it was required to use another factor 5 for the authors' expression, to take best advantage of the chart space. The other two factors were suggested by the writer as more nearly expressive of the probable influence of pondage. It was also necessary to divide the quantities of the factor involving *s* in Eq. 6*d* by 4, to keep the right-hand curve within the limits of the drawing. The outstanding characteristics of curves relating to length of runoff travel, drainage area, and altitude above the outlet are increased curvature and divergence above 100.

3. The general form of Eqs. 6 with a fractional power of drainage area in the numerator and with a lesser power of runoff travel distance in the denominator seems to recall the derivation of the width[24] formula derived by C. R. Pettis, M. Am. Soc. C. E., directly from the Modified Myers formula,[4] Eq. 1. Through the introduction of a "compactness factor" Colonel Pettis obtained the expression,

$$Q = 328\,P\,W^{1.25} \qquad (7)$$

in which P represents the rainfall depth in inches for the critical or adopted period and W is the average width of the drainage basin in miles, or $\frac{M}{L}$.[25] (A review and summary of Colonel Pettis' paper is readily available.)

4. Within the ordinary range of observed drainage areas, where length is some multiple of the width, it is possible to convert the authors' fractional power of M divided by a lesser power of L to a power of W quite closely approaching that used by Colonel Pettis, with some advantage in the way of simplification, but with appropriate changes in coefficients.

[24] "A New Theory of River Flood Flow," by C. R. Pettis, 1927 (privately published).

[25] "Floods in the United States; Magnitude and Frequency," *Water-Supply Paper No. 771*, U. S. Geological Survey, 1936, pp. 36–37.

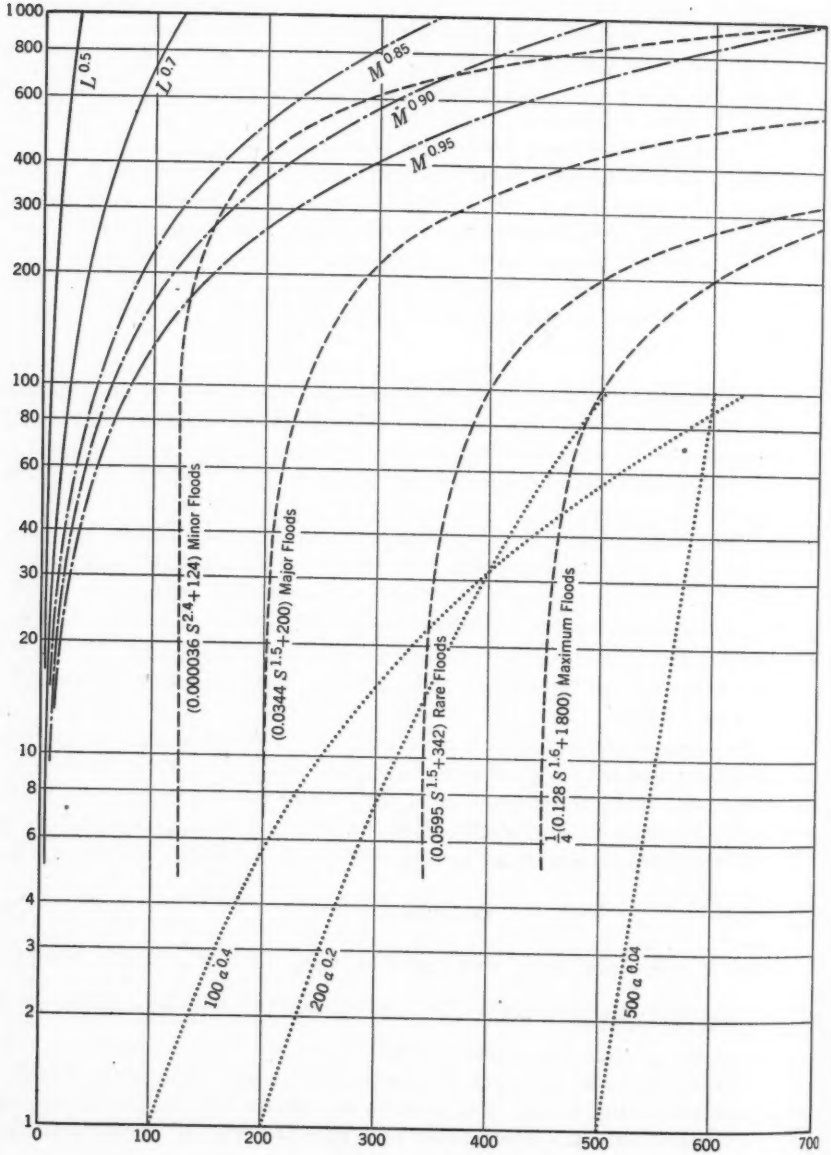


FIG. 14.—FLOOD FORMULA FACTORS

5. With 681 station-years of record included for forty-eight items of Table 1, giving an average of 14.2 years per station and a maximum of 30 years per station, and with only four of the observed maxima exceeding the quantities listed in Col. 11 as "rare" (the amount of such excess being 25% or less), it is difficult to reconcile the idea of adding another 300% or more to attain the "maximum flood" as determined by the authors. Addition of 150% might well suffice.

6. Critically examining the various factors of Eqs. 6, graphically outlined in Fig. 14 (with such alterations as seemed essential to keep them within limits of the drawing), the following comments are submitted—

(*a*) The values of $L^{0.7}$ vary between 3.09 and 15.5 for 5 miles and 50 miles, respectively, or 77.5 for 500 miles. Thus, as the L-factor appears in the denominator of each formula, its influence would be such as to decrease flood peaks to 1/25 or 4% for such a range of variation. For the alternate Eq. 6*b*, with $L^{0.5}$ instead, the total range between 5 and 500 would be 10 instead of 25—a more readily defensible factor according to extensive records and observations.

(*b*) The s-factor for minor floods, with added constant, amounts to 126.27 for 100 ft of altitude, or 137 for 200 ft, 237 for 500 ft, or 694 for 1,000 ft of altitude above the outlet. Thus, because of the influence of altitude in Eq. 6*a*, the flood peak is multiplied by a factor 5, in consequence of increased median altitude from 200 to 1,000 ft. All three of the other s-factors and associated constants run off the drawing before attaining an altitude of 600 ft. Their trends magnify the influence of altitude of drainage area above the outlet in a greater degree at 1,000 ft than has been noted for minor floods.

(*c*) The pondage factor, a, as used by the authors shows but slight variation between 1% and 5%, attaining a value of only 1.067 or 1.169 at 50%. In other words, the regulation afforded by ponds covering fully half of the drainage area would decrease the peak to 1/1.169 or 85.6% of its magnitude with no pondage. If the fractional power were 0.4 instead of 0.04, the corresponding values of the a-factor would become 1.90, 3.90, and 1/3.90 or 25.7%. This is more nearly in accord with the writer's experience, although 0.2 might serve somewhat better, with values of 1.38, 2.19, and 1/2.19 or 45.8%.

The proper scope of the authors' equations and formulas in this paper lies mainly if not entirely below 1,000 for both M and s—drainage area and median altitude above the outlet, respectively. Similarly, the width formula derived by Colonel Pettis was intended to serve primarily for drainage areas ranging from 1,000 to 10,000 sq miles; it occasionally served fairly well outside those limits, but more often wandered beyond control.

For many years the writer has striven for simplification of expression within the hydrologic field. One of the prime essentials seemed to be a formula

capable of comparing the floods of either the Missisippi River or the Amazon River with the peaks of torrential floods at their headwaters, regardless of areal extent. Investigations covering several years led to a measure of approval and acceptance by the hydrologic and engineering profession, as demonstrated by a wide use of the square root of drainage area as a convenient first approximation to the influence of areal extent for the entire range above 4 to 7 sq miles. Essentially the same conclusion was reached as the result of a later investigation conducted by a committee of the Boston Society of Civil Engineers. This report laid some of the foundations for later development in the field of unit hydrographs, as has been tacitly acknowledged by the originators of that procedure.[26] For the usual form of drainage area, with length considerably greater than average width, introduction of the expression $\sqrt{M}$ may be regarded as equivalent to adopting the width as the governing dimension with either a coefficient or a power greater than unity; or it might be construed as being the same as using the dimension of length in the denominator, with either a coefficient or a power somewhat less than 1. These developments have been presented in considerable detail in some of the references in the paper, as well as in other publications deserving mention.[27,28,29,30,31,32,33]

In spite of the time-consuming mazes of complex computations involved in the authors' procedure and formulas, not to mention the occasional exceptions or divergences of opinion freely expressed by the writer as constructive suggestions, the entire investigation, together with its lucid presentation, is both profitable and enlightening to the profession. The concepts of basin characteristics for not only flood phenomena but also rainfall and runoff relations generally are not new. The late Emil Kuichling, M. Am. Soc. C. E., the late A. N. Talbot, Past-President and Hon. M. Am. Soc. C. E., Major Myers, and many other authors of formulas made special provision for accomodating the various factors throughout their probable range. The writer's modification of the Myers formula included extension of the range both above and below that originally suggested by Major Myers. The use of the Modified Myers formula is so simple and elementary as to deter its adoption by those who presumably seek the more abstruse and complicated forms. No matter how involved any preferred procedures, the results are readily translated into terms of Myers percentage ratings, as demonstrated by Table 2.

[26] Report of Committee on Floods, *Journal*, Boston Soc. of Civ. Engrs., September, 1930, pp. 297 and 376–407.

[27] "Methods of Estimating Flood Discharge," by Robert L. Lowry, *Bulletin*, Colorado Soc. of Engrs., June, 1936.

[28] "Handbook of Applied Hydraulics," Calvin V. Davis, Editor, McGraw-Hill Book Co., Inc., New York and London, 1st Ed., 1942, Sections 1 to 3, pp. 1–132.

[29] "Maximum Stream-Flow with Reference to Flood-Formulas," by C. S. Jarvis, *Transactions*, Am. Geophysical Union, 1937, pp. 409–419.

[30] "Great Floods in the United States," by C. S. Jarvis, *ibid.*, 1939, pp. 157–166.

[31] "Estimating Annual Yield from Ungaged Drainage-Basins and Effect of Land-Use Treatment on Surface Runoff," by C. S. Jarvis, *ibid.*, 1941, pp. 875–893.

[32] "Derivation of River Discharge from Measurements of Tributaries," by C. S. Jarvis, *ibid.*, 1942, pp. 517–533.

[33] "Gleanings from the Field of Hydrology," by C. S. Jarvis, *ibid.*, 1943, pp. 651–693.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FAILURE OF THE TACOMA NARROWS BRIDGE

REPORT OF THE SPECIAL COMMITTEE OF THE BOARD OF DIRECTION

Discussion

BY HOMER M. HADLEY

HOMER M. HADLEY,[3] ASSOC. M. AM. SOC. C. E.[3a]—Some additional comment may well be made on the depth-to-span ratio of the late Tacoma Narrows Bridge (1 : 350) which in conjunction with its narrow breadth made it a structure of extreme slenderness and flexibility. A number of places in the report of the Special Committee of the Board of Direction and its appendixes refer to this $\frac{d}{l}$-ratio. In the minds of many people it seemingly constitutes a damning indictment of the bridge and in itself the full and sufficient cause of the disaster. Therefore, it appears desirable to direct attention to certain other structures which do not conform to this criterion but stand in denial to it. Such a criterion, properly limited and restricted, may well be accepted by every one—in its field of application. Great is the accomplished fact. The bridges that have been built and stand and serve their purpose fully and well should be given the most thoughtful and respectful consideration. Reality is in them, and reality is greater than theory—greater than reams of mathematical calculations, greater than laboratory experiments, no matter how technical and elaborate. Because this is so is, however, no warrant for arbitrarily taking one aspect of reality and declaring that it is the essence of the whole.

Bridge 2 in Table 1 of Report (*a*) is none other than the George Washington Bridge at New York, N. Y. Various data concerning it are given and its depth-to-span ratio is recorded as 1 : 120. A footnote corrects the tabulated dead weight and the number-of-track values for "the present single-deck con-

NOTE.—This report was published in December, 1943, *Proceedings*. Discussion on this report has appeared in *Proceedings*, as follows: April, 1944, by Francis P. Witmer.

[3] Regional Structural Engr., Portland Cement Assn., Seattle, Wash.

[3a] Received by the Secretary May 4, 1944.

dition" but fails to correct the depth-to-span ratio for this condition. Footnote (b), Table 1, states that "depth" in Col. 8 is "depth of stiffening girder." Accepting this as the value to be used in "the present single-deck condition" the depth-to-span ratio is found to be about 1 : 1300, since the stiffening girders are no more than the top chords of the future trusses which will be built when the lower deck is added. Even if the greatest possible combination of longitudinal stiffening members is used—stringers, fascia girders, etc.—the ratio is still 1 : 350, the same as that of the ill-fated Tacoma Narrows Bridge.

Undoubtedly minor movements occur in the deck of the George Washington Bridge but so far as the writer has ever heard these movements are very minor, in spite of the open exposure to wind and storm. Why should this condition be, if a fairly low $\frac{d}{l}$-ratio is essential for stability? The great weight of the deck and of the four 36-in. cables is one explanation offered. Another is of course the fact that there is very little projection in this deck for a horizontal wind to impinge on—little cause for wind-stream disturbances across its upper surface; but whatever the cause of the stability the fact remains that here is demonstrated the inadequacy of any broad and sweeping generalization that would seek to establish certain $\frac{d}{l}$-values as criteria for judging the sufficiency of design.

Another example is the Ronosfoss Bridge in Norway with a center span of 600 ft and a $\frac{d}{l}$-ratio of 1 : 458. Nevertheless, according to Louis Balog[4] "* * * no movement of this bridge due to wind was reported, and it carries traffic satisfactorily."

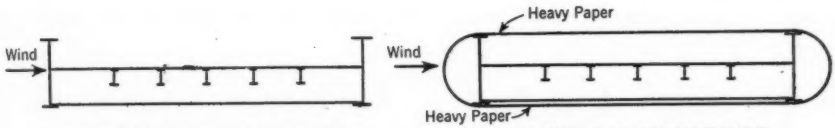
A very remarkable suspension bridge—the design of which is a truly fine example of engineering skill and judgment in its adaptation to the site and to the need for extreme economy—is the Royal Gorge Bridge in Colorado.[5] It crosses the canyon of the Arkansas River, 1,053 ft above the tracks of the Denver and Rio Grande Western Railway. Its main span, 880 ft, is not much less than that of the Peace River Bridge on the Alcan Highway in British Columbia, Canada, completed in 1943. "The floor system, which is suspended from the cables at 10-ft. panel points by 1⅛-in. steel suspender rods, consists of 15-in. floorbeams, which in turn support seven lines of 8-in. I-beams and two lines of 8-in. channel stringers."[6] This construction yields a maximum $\frac{d}{l}$-ratio of 1 : 1320. There are no stiffening trusses or girders. However, by the use of wind cables, stability is obtained. Wind cables are evidently a limitation on the $\frac{d}{l}$-criterion.

Still another limitation is that of the basic shape and form of the deck itself and of the resistance offered to the passage of wind around it. There should be no question about the importance and reality of this fact in any

[4] *Engineering News-Record*, December 16, 1943, pp. 74.
[5] *Ibid.*, June 5, 1930, p. 922.
[6] *Ibid.*, p. 924.

one's mind. The analogous condition of the flow of water around bridge piers and the effect of pier shape is freely recognized by all.[7] The effect of shape is shown in the airplane wing. Time was when the biplane was the dominant form, with its upper and lower "planes" trussed together by struts and wires. Where is the biplane today? It is practically obsolete and it has been replaced by the monoplane not because the biplane was structurally inadequate but because the monoplane is much better suited to its purpose.

In the experimental work with the 100-ft-long model of the Tacoma Narrows Bridge jointly sponsored by the University of Washington, the Washington Toll Bridge Authority, and the U. S. Public Roads Administration, Professor Farquharson found that the scale-replica model of the open steelwork of the deck that failed (that is, 8-ft side girders, floor beams, stringers, wind trusses, etc., without floor slabs, sidewalks, or diagonal stays at midspan) was extremely sensitive and flexible in winds of low velocity (see Fig. 12(*a*)). It



(*a*) APPROACHING DESTRUCTION IN 30-MILE WIND (*b*) STABLE IN 125-MILE WIND

FIG. 12

developed more pronounced movements than the prototype and in a 30-mile wind, also of laboratory scale, whipped so violently that the test had to be terminated. Following this the same identical frame was given a paper encasement and was provided with semicircular nosing on both sides, as shown in Fig. 12(*b*). When this model was exposed to the 30-mile wind, there was no movement. It hung inert. Vigorous undulatory and torsional motion was imparted to the model and, as soon as hands were removed, the motion damped out and it again hung still. In all winds up to 125 miles per hr the behavior was the same with the exception that in the 125-mile wind the imparted motion continued. The $\frac{d}{l}$-ratio did not change. The moment of inertia of the deck did not change. The cables did not change. What had changed, however, was the flow of the wind stream around the deck; and by that action a very wonderful change was effected in wind loads. Tests along these particular lines have not been extensive but sufficient work has been done to indicate the revolutionary possibilities inherent in the situation if shape and form can be substituted for $\frac{d}{l}$-ratios and great weight.

Report (*a*) states "* * * there is no doubt that sufficient knowledge and experience exist to permit the safe design of a suspension bridge of any practicable span." What is this "practicable span"? It is one thing if current convention must be adhered to; it is another thing if it can be revised and

[7] "Bridge Piers as Channel Obstructions," by David L. Yarnell, *Technical Bulletin No. 442*, U.S.D.A., November, 1934.

simplified. It is one thing with a lavish economy; it is another thing if lavishness must end. Certainly no one can gainsay the fact that long-span suspension bridges do entail heavy costs. Neither can it be gainsaid that there are numerous places in the United States where, due to navigation requirements, depth of water, limited traffic, etc., narrow long-span bridges are badly needed but are not "practicable" under existing standards. The Tacoma Narrows site was such a place. It was in an effort to meet the very real and urgent necessities of the project at lowest possible cost that its design was developed. Unfortunately the roadway was placed 3 ft below the top of the girders with the resultant turmoil and turbulence in the wind stream which has revealed itself in every bridge with I-shaped stiffening members. If, however, with necessary readjustments of framing, the roadway had been placed at the top of the girders, the section had been closed with plywood at the bottom of the girders, and fairing in the form of plywood nosing had been added at the sides, it appears very probable that the bridge which failed so disastrously would have been standing today, giving 100% satisfactory service without the slightest change in its width or in its $\frac{d}{l}$-ratio.

Every one is wiser after the event. What is apparent today was far from evident yesterday. The dramatic failure of the Tacoma Narrows Bridge has a profound message for all engineers, but it imposes no necessity to adhere rigidly to past practices. Its lesson is no other than this: Whatever you do must be right with Nature or woe unto you!

A bit of high school Latin absurdly clings in the writer's memory when more important things are forgotten—two lines from Ovid concerning Phaëthon, the bold youth who essayed to drive the sun chariot of his father, Apollo, across the sky. The wild horses of the chariot broke from his control and dashed headlong on their course, careening up and down, scorching earth and heaven. At length Jove was compelled to strike Phaëthon down with a thunderbolt and he fell to earth. The water nymphs gathered up his remains, buried him, and placed a stone at his head on which they inscribed these lines:

> "Hic situs est Phaëthon, currus auriga paterni.
> Quem si non tenuit, magnis tamen excidit ausis."

Freely translated they are:

> "Here lies Phaëthon, driver of his father's chariot.
> Although he was unsuccessful, he failed in a great undertaking."

Of the first Tacoma Narrows Bridge let it be remembered that its failure was failure in a great undertaking.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ANALYSIS OF THE GENERAL TWO-DIMENSIONAL FRAMEWORK

Discussion

BY LEON BESKIN

LEON BESKIN,[8] ASSOC. M. AM. SOC. C. E.[8a]—To clarify this discussion, the method of framework analysis presented by the author is summarized as follows:

1.—The redundant unknowns are divided into two groups—(*a*) linear displacements of the joints; and (*b*) rotations at the joints, for which the moments at the joints can be substituted in the case of relatively simple structures.

2.—Group (*a*), Item 1, is represented by the relative variations of lengths (strains) of the sides of a statically determinate triangulated system which "maps" the actual structure according to a method well known in surveying.

3.—Group (*b*), Item 1, can be calculated in a relatively simple manner as a function of group (*a*) when the framework is of the "open" type; that is, when the computer progresses along the framework in a definite direction without intersecting hinges (free rotation) or fully restrained multiple supports (no rotation), no member can be encountered twice. In this special case, moments can be considered directly, instead of rotations. The procedure becomes increasingly complicated when multiply-connected frames are considered and it becomes necessary to introduce a third group of redundant unknowns, which must be added to the moments and which includes the rotations of some of the joints.

4.—The equilibrium of moments at the joints is obtained without considering the equilibrium of the longitudinal and transverse forces. These forces can be defined in terms of the moments at the joints. Then the structure can be replaced by an equivalent hinged structure and the conditions of stability and equilibrium of forces corresponding to that structure give the equations necessary to define the unknowns of group (*a*), Item 1.

NOTE.—This paper by Yves Nubar, M. Am. Soc. C. E., was published in April, 1944, *Proceedings*.

[8] Structures Engr., Consolidated Vultee Aircraft Corp., Allentown, Pa.

[8a] Received by the Secretary May 3, 1944.

A profitable comparison can be made between this paper and the simple and direct analysis presented by David M. Wilson,[9] M. Am. Soc. C. E., who admits "that no new method of analysis is proposed." For example:

(1) Divisions of the unknowns into two groups is the basis of both methods.

(2) For group (*a*), Item 1, Mr. Wilson uses rotations and Mr. Nubar uses strains. The two groups are similar, and only practical considerations can indicate the superiority of one group. Mr. Wilson limits himself to structures in which axial deformations are negligible, which is the "less general case" treated by Mr. Nubar; but the treatment of the more general case by successive approximations is well known; it is the conventional method for the analysis of trusses. A good example can be found in a study by K. A. Calisev,[10] who examines trusses, beams on continuous supports, and frameworks.

(3) For group (*b*), Item 1, Mr. Wilson uses rotations or moments, and applies slope deflection and moment distribution methods. Mr. Nubar uses moments wherever possible and applies the method of fixed (conjugate) points. All these methods are well known and the special conditions of each problem or, more often, the habits of the analyst, dictate the choice between these methods. However, for multiply-connected circuits, the moment distribution method has the definite advantage of being well developed, available with numerous examples in any textbook. On the contrary, the fixed points method, developed many years ago in a straightforward manner for open circuits only, is far from being perfected (as shown by the author's statements) for use in analyzing multiply-connected circuits. The author indicates, in substance, that some rotations must be introduced as auxiliary unknowns and that the system of equations containing these unknowns must then be solved. This is known and is a condition that must be applied to any statically indeterminate structure. A method is new and useful essentially in the extent to which it presents a definite and straightforward manner of choosing unknowns and of solving systems with a high degree of indeterminacy.

(4) The determination of group (*a*), Item 1, when group (*b*) is defined in terms of group (*a*), is identical in the Nubar and the Wilson methods.

From the foregoing, it appears that the Nubar method follows closely the pattern of conventional methods. It remains to be decided whether the method introduces an element of newness and simplification over conventional procedures.

As indicated, the division of unknowns in two groups is conventional. Generally the unknowns of group (*a*) are: The horizontal displacements for multi-story frames; the vertical displacements for Vierendeel beams; and the axial loads (or strains) for triangulated trusses.

The author replaces unknowns that result from a choice adapted to the nature of each problem, by a system of arbitrary unknowns which are the same in all cases. This attitude can be compared to that of an analyst who would always

[9] "Analysis of Rigid Frames by Superposition," by David M. Wilson, *Proceedings*, Am. Soc. C. E., February, 1944, p. 129.

[10] "The Method of Successive Approximation for the Analysis of Statically Undetermined Systems," by K. A. Calisev, *Publications*, International Assn. for Bridge & Structural Eng., Vol. IV, 1936, Zurich, p. 199.

use rectangular coordinates, even if other sets of coordinates (polar, cylindrical, spherical, etc.) are more adapted to the problem. Such an approach is utterly unsatisfactory; and sometimes the main difficulty in the solution of a problem is in the choice of a proper system of coordinates. In the present case, since it is always necessary to calculate the angular displacements δ of the straight lines connecting the ends of each member (called unit deflections by the author), it appears logical, wherever possible, to use the latter as principal unknowns. For instance, in Fig. 13 no advantage results from the use of ϵ_6 as an unknown, against the use of either the horizontal displacement of AB or the common rotation of EA, BF, and CD. Defining δ as this rotation, Eqs. 36, 37, and 38 become:

$$\left.\begin{aligned} -\delta_{8-4} &= -\delta_{1-5} = \delta_{3-9} = \delta_{4-1} = \delta_{5-8} = \delta \\ \delta_{1-2} &= \frac{16}{9}\delta = 1.777\,\delta \\ \delta_{5-2} &= -\delta_{2-3} = (\delta_{3-9} + \delta_{1-2}) = 2.777\,\delta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (49)$$

The simplification is obvious and clearly demonstrates, by means of the example given by the author himself, that the method is of very little practical interest for frameworks, even of an unconventional type.

For a conventional framework, there is no doubt that the linear displacements of the various stories, defining the rotations at a glance or (even better) the rotations of each line of columns between stories, are the logical unknowns to be used (Fig. 14). If axial strains are considered, they are added to the rotations to form the first group of unknowns.

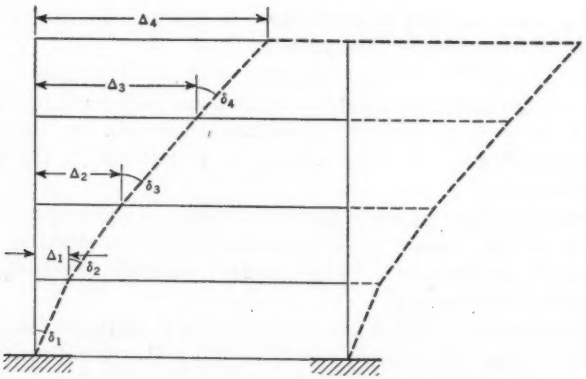


FIG. 14

His illogical choice of the unknowns for rectangular frames has forced the author to use, in all cases, what he called "equations of compatibility" which relate axial strains to angular displacements. These equations are well known under other names. Eqs. 4 are the equivalent of the Williot-Mohr construction. The condition expressed in terms of moments (Eq. 5) is the equation used for the calculation of deflections of curve beams, the sole dif-

ference being that the quantity $\sum x_k (\epsilon_{k,k+1} - \epsilon_{k-1,k})$ is replaced by the equivalent quantity $\sum \epsilon_{k,k+1} (x_{k+1} - x_k) = \sum \frac{P}{E A} (x_{k+1} - x_k)$, whereas the quantity $\sum y_k \left(\frac{\Delta_{k,k-1}}{L_{k,k-1}} - \frac{\Delta_{k-1,k}}{L_{k-1,k}} \right)$ is generally replaced by the integrated expression:

$$\int y_k \frac{d^2\Delta}{ds^2} ds = \int y \frac{M}{E I} ds \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

Eq. 10*a* is well known, in English[11] as well as in French[12] literatures under the equivalent form (see Fig. 15):

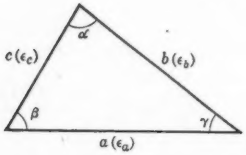


FIG. 15

$$\frac{\Delta_b}{b} - \frac{\Delta_c}{c} = \delta \alpha = \frac{\epsilon_b - \epsilon_a}{\tan \gamma} - \frac{\epsilon_c - \epsilon_a}{\tan \beta} \ldots\ldots (51)$$

Topology of Frameworks.—The purpose of the section entitled "Topology of Frameworks" appears to be the establishment of Proposition (*c*) which appears several times in the paper. However, this proposition is self-evident without any preliminary proof: The n_p-joints each have two degrees of freedom in the plane; the entire framework in joint displacement has $2\,n_p$ degrees of freedom, plus n_σ degrees of the n_σ linear displacements of support, minus $n^r{}_m$ degrees for the $n^r{}_m$ rigid (nondeformable) members. Thus:

$$\text{Degree of freedom} = 2\,n_p + n_\sigma - n^r{}_m \ldots\ldots\ldots\ldots\ldots (52)$$

In Eq. 52, $n^r{}_m$ (in which r is an index and not an exponent) may vary from zero to n_m, the total number of members. It appears that this entire section is extraneous to the subject of the paper.

Boundary Conditions.—Since moments and axial strains are used as unknowns, and, perforce, they define the stresses and loads in the structure completely, it is obvious that the boundary conditions can be expressed in terms of such elements, and general Eqs. 24 do not clarify the problem for practical applications.

End Moments.—The method used to determine in succession the moments in a structure of the "open" type is the well-known method of "conjugate points" (English terminology), "focal points" (French terminology), "fixed points" (German terminology).

The development of Eqs. 35 can be presented in a simpler and more symmetric manner: Using the author's notation, and with r denoting any member, it can be shown that, if n members meet at a joint J and if, for $n - 1$ members, there exist relations of the type

$$M_{rR} + C_{rJ} M_{rJ} = D_{rJ} \ldots\ldots\ldots\ldots\ldots\ldots (53)$$

(R being the joint at the end of the member r), a relation of the same type holds for the nth member.

[11] "Airplane Structures," by A. S. Niles and J. S. Newell, Vol. I, 3d Ed., 1943, New York, N. Y., p. 402.

[12] "Résistance des Matériaux," by B. de Fontviolant, Vol. II, 1927, Paris, p. 148.

This fact is obvious from the general relations of linearity between moments and rotations. The coefficients C_{n-J}, D_{n-J} of the nth member can be related to the other coefficients. The equation of equilibrium of a joint is:

$$\sum(M_{rJ} + M_J) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

in which M_J is a couple applied at the joint. Expressing the rotation of any member at the joint in terms of the loads and couples,

$$\omega_J = \delta_r + \frac{1}{6\,E\,k_r}(2\,m_{rJ} - m_{rR} + M_{rR} - 2\,M_{rJ}) \ldots\ldots\ldots (55)$$

All rotations ω_J are equal. Replacing M_{rR} in terms of M_{rJ} by means of Eq. 53, the value of M_{rJ} can be expressed as follows:

$$M_{rJ} = -\,\omega_J \frac{6\,E\,k_r}{C_r + 2} + \frac{6\,E\,k_r\,\delta_r + 2\,m_{rJ} - m_{rR} + D_{rJ}}{C_r + 2} \ldots\ldots (56)$$

Substituting in Eq. 54:

$$M_J + \sum\frac{6\,E\,k_r\,\delta_r + 2\,m_{rJ} - m_{rR} + D_{rJ}}{C_r + 2} - \omega_J\,6\,E\sum\frac{k_r}{C_r + 2} = 0 \ldots (57)$$

When M_J, δ_r, m_r, and D_r are all equal to zero, there may be a rotation ω_J under the action of moments M_{rR}. This shows that the coefficient of ω_J is equal to zero (except for a completely restrained joint for which $\omega_J = 0$ and $C_r = -2$). Thus

$$\sum\frac{k_r}{C_r + 2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

Also

$$M_J + \sum\frac{6\,E\,k_r\,\delta_r + 2\,m_{rJ} - m_{rR} + D_{rJ}}{C_r + 2} = 0 \ldots\ldots\ldots (59)$$

Eqs. 58 and 59 are absolutely symmetrical and are equivalent to Eqs. 34 and 35, provided it is noted that $C_{rR} = 1{:}C_{rJ}$ and $D_{rR} = D_{rJ}{:}C_{rJ}$. The method developed from Eqs. 58 and 59 is a mere extension of the method of fixed points for continuous beams, adapted to "open" structures. In that form it was presented by Prof. M. Ritter[13] as early as 1936.

For multi-connected circuits, it is necessary to introduce the rotations of some of the joints, which cannot be eliminated directly. Very rapidly, the method becomes extremely complicated, due to the number of simultaneous unknowns, and loses its value, at least in its present stage. Moreover, the author limits himself to a simple suggestion of the method, without attempting to develop it into a practical procedure. For a conventional truss, the number of rotations to be introduced is the integer equal or immediately superior to one third of the number of the circuits. In Fig. 16(*a*), the "opened" joints are represented by hinges. For a conventional Vierendeel structure (Figs. 16(*b*) and 16(*c*)), the corresponding ratio is one half. The last result also applies to a one-span, multi-story portal, and to a two-story, multi-span portal (Fig. 16(*d*)). For a multi-span, multi-story portal, the number will

[13] "Continuous Beams on Compressible Supports," by M. Ritter, *Publications*, International Assn. for Bridge & Structural Eng., Vol. IV, 1936, Zurich, p. 509.

be approximately two thirds to one half of the order of multiplicity of the portal. For instance, the portal represented in Fig. 16(*e*) is of 42d order. It can be reduced to the 6th order as shown, and then to the "open" type as shown in Fig. 16(*f*). Thus a total of twenty-eight rotations are needed for a framework of the 42d order, and four rotations are required for a framework of the 6th order. It is obvious that in such cases only iteration methods are of practical interest, and in these methods it may be advantageous to use all the rotations of joints in a symmetric manner, avoiding the separation of any that may be considered as fundamental unknowns in order to calculate the remainder in terms of the segregated unknowns.

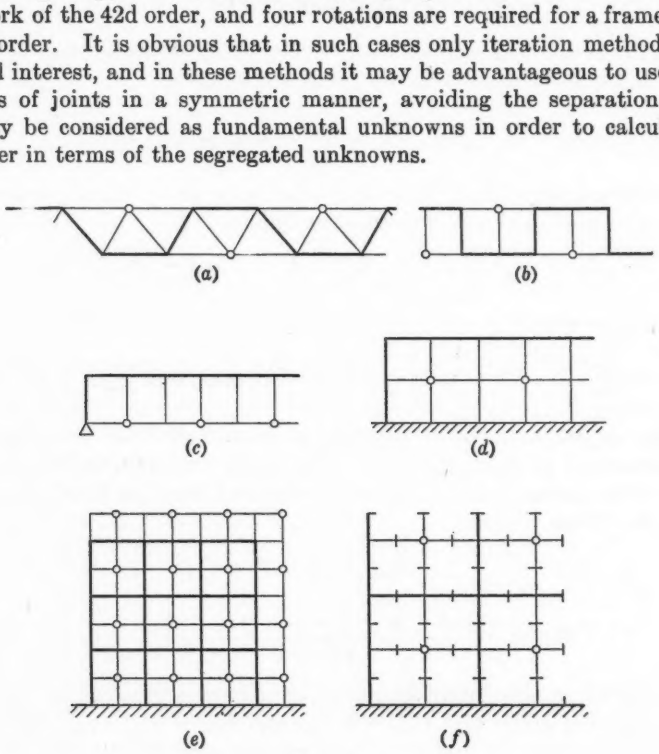


FIG. 16

When the author states (see heading, "Determination of End Moments") that "The proposition will be established that the end moments in the members of a rigid frame can be found as linear functions of the δ_{p-q} elements * * * and the linear movements of the supports," he simply states that to solve a system of linear equations the following procedure can be followed:

Solve the equations in terms of the given constants, some of the unknowns being considered as fundamental unknowns. The remaining unknowns will be expressed as linear functions of the fundamental unknowns and the constants. Thus, the author's statement, presented as a new proposition, does not reflect any new light on the problem. Moreover, the proposition is incomplete, since the end moments are functions, not only of the δ-elements, but also of the rotations that must be introduced in multi-connected circuits. However, for "open" structures, that procedure, which could be designated as Ritter's method, is of great interest because: (1) The number of fundamental

variables is small; and (2) the secondary variables can be expressed in a relatively simple manner in terms of the fundamental variables.

If similar methods were available for multi-connected circuits, a great step would be made toward the general direct analysis of structures of that type. It appears that such methods do not yet exist.

Determination of the Axial Stresses and Linear Movements of the Supports.—As indicated by Mr. Wilson,[9] the conventional procedure is to consider appropriate free bodies and to determine the transverse forces in the members. The conditions of stability and equilibrium of the totally hinged structure, or the conditions of equilibrium of several free bodies that are statically independent of one another, define all the conditions required to find the fundamental unknowns. For instance, in the simple case of the frame of Fig. 17, the hinged structure is unstable. Thus, the equilibrium of two free bodies must be examined: The condition of equilibrium of body *I* defines the load acting on body *II*. Then the equilibrium of body *II* can be expressed, defining the load acting on the vertical member.

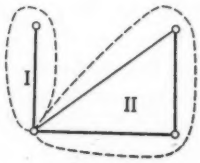


FIG. 17

In order to solve the same problem, the author introduces a fictitious diagonal and stipulates that the load in that diagonal must be equal to zero, which appears as a useless complication. Actually, the author, in Fig. 13, does not use that diagonal, and his method, somewhat artificial, does not give the required equation directly. It is possible to find the result directly. For

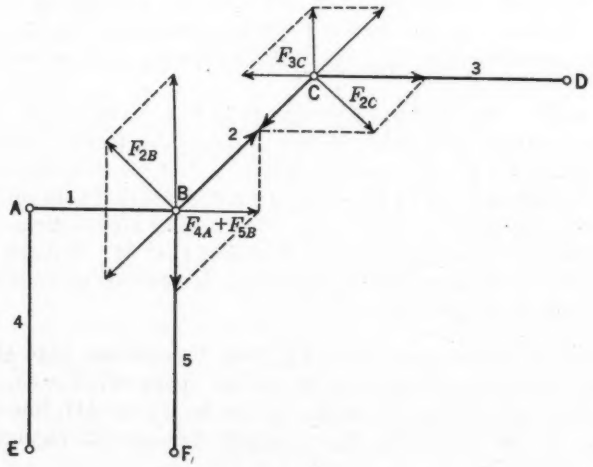


FIG. 18

example, projecting the forces as shown in Fig. 18, in order to eliminate the loads along 3 and 5:

$$F_{2B} + F_{2C} - (F_{4A} + F_{5B} + F_{3C})\sqrt{2} = 0 \ldots\ldots\ldots\ldots (60)$$

which is strictly equivalent to Eq. 47 after the subsequent elimination of T_4 from the latter equation.

Conclusions.—The following conclusions deserve emphasis:

1.—The author does not present a substantially new method of analysis of redundant structures; he merely gives a somewhat different presentation of existing methods. The presentation shows a tendency to generalize the field of application of some methods. In the case presented, this approach is erroneous, since absolutely general methods are available (least work). The trend of future research must be toward the development of special methods well adapted to each category of problems, since absolutely general methods are impracticable in highly redundant structures, because of long numerical computations.

2.—The author introduces conditions that create undue complications in the presentation of simple and well-known processes, such as his introduction of conjugate frames and his treatment of the subject "Topology of Frameworks." He also presents well-known concepts with a changed and special terminology, such as compatibility equations, boundary conditions, the incomplete proposition relative to the determination of end moments as linear functions of the δ_{p-q}-elements, and the fundamental relation of the fixed-point method.

3.—Precisely at the point where the method seems to suggest new developments (in the introduction of a minimum of additional rotations), the author merely states that these quantities must be introduced and then eliminated.

4.—An incorrect interpretation is given to the number of the principal unknowns. Actually, to the unknowns defining the joint displacements must be added the unknowns defining the joint rotations in multi-connected frameworks.

5.—The author has repeated a summary in two places of his paper—first in the "Introduction" and again in the "Application to the General Case." The second summary is much more intelligible than the first but, when compared with the outline given by Mr. Wilson,[14] it appears that both summaries are practically identical step by step, as far as principles are concerned. From a practical standpoint, however, there is no doubt that Mr. Nubar's paper will prove much less valuable to the engineer who is desirous of finding a simple and rapidly applicable method.

Correction for *Transactions:* In April, 1944, *Proceedings*, page 447, line 28, change "they will be" to "coefficients D will be"; page 447, line 38, change "C and D will also be" to "coefficients D will also be"; page 447, line 46, change "all of them" to "all coefficients D"; and, page 449, line 12, change "m_{mt}" to "n_{mt}."

[14] "Analysis of Rigid Frames by Superposition," by David M. Wilson, *Proceedings*, Am. Soc. C. E., February, 1944, p. 131.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

GEOLOGY IN HIGHWAY ENGINEERING

Discussion

BY R. WOODWARD MOORE, GEORGE E. EKBLAW, AND ALLEN S. CARY AND ADE E. JASKAR

R. WOODWARD MOORE,[15] ESQ.[15a]—The twelve cases discussed in this paper constitute important problems continually confronting the highway engineer wherever road construction and road maintenance are undertaken. The writer was particularly interested in the references to geophysical methods of test applied to their solution. The possible value of such exploration methods to the geologist and engineer alike should be emphasized.

For some years the Division of Physical Research and Tests of the Public Roads Administration has been engaged in research on the use of geophysical methods of test in highway engineering. Conclusions resulting from these studies and the work of several of the state highway departments, as well as of other agencies in the United States and elsewhere, indicate that the methods of test mentioned by the author—the earth-resistivity and the refraction seismic tests—may have considerable economic value in the solution of many of the problems listed by Mr. Huntting. Specifically, Problem (1) and Problems (5) to (10), inclusive, appear to be susceptible of solution or partial solution by geophysical tests. The following paragraphs are submitted in amplification of the information concerning geophysical tests as presented by the author in the belief that such methods of subsurface exploration may prove to be increasingly useful to the geologist in further demonstrating the usefulness of geology in highway engineering.

Problem (1).—The location and delineation of deposits of road-surfacing materials by earth-resistivity tests have been described by various investigators during the ten years since 1934 (50)(51)(52).[15b] In a letter to the

NOTE.—This paper by Marshall T. Huntting was published in December, 1943, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1944, by W. W. Crosby; March, 1944, by Carl B. Brown; April, 1944, by F. H. Kellogg, Berlen C. Moneymaker, E. F. Bean, A. T. Bleck, Lyman W. Wood, Philip Keene, and Jacob Feld; and May, 1944, by H. E. Marshall, K. B. Woods, and D. J. Belcher.

[15] Associate Civ. Engr., Public Roads Administration, Federal Works Agency, Washington, D. C.

[15a] Received by the Secretary April 25, 1944.

[15b] Numerals in parentheses, thus: (50), refer to corresponding items in the Bibliography (see Appendix of the paper), and at the end of discussion in this issue.

writer, Henry C. Porter, M. Am. Soc. C. E., reported rather favorably on such tests by the State Highway Department of Texas. Investigations conducted by the Public Roads Administration have further established the usefulness of this test in such work.

Problem (*5*).—Both the refraction seismic test and the earth-resistivity test have been found to be applicable to the classification of excavation materials (53). The resistivity test has been applied rather successfully in the study of highway grading projects by the Missouri State Highway Department, the test data obtained being of considerable aid in predicting the character of the materials to be encountered (54).

Problem (*6*).—Although little work has been done, as yet, in determining the usefulness of geophysical tests in connection with landslide areas, it would seem highly probable that such tests could be of appreciable value in disclosing potential slide areas. The refraction seismic test has been used in preliminary investigations to determine the presence of impervious shales beneath talus slopes which constitute potential sliding surfaces upon which the overlying porous material may move (55). The earth-resistivity test should be of practical value in locating impervious strata of shale or clay beneath less consolidated and more porous soils. Where such conditions exist, road structures may be benched into the impervious layer instead of being built entirely upon the porous materials subject to slipping along the lubricated surface of the underlying impervious stratum. In the case of large fill structures in highway construction routine resistivity tests made over the fill immediately after construction and at intervals thereafter may prove of value in disclosing variations in moisture content. In this way abnormal increases in moisture may be detected within the fill where increasing hydraulic pressures may cause a slip-out requiring costly reconstruction work and causing considerable inconvenience to an established flow of traffic. No data are at hand to substantiate such a conjecture, however.

Problem (*7*).—Both methods of test may be used in the study of bridge foundation materials, the seismic test being more positive in its identification of solid rock formations in areas where either solid rock or gravel formations may be expected. The low cost involved and the rapidity with which these tests can be executed make them of particular value in preliminary surveys, often eliminating the need for the time-consuming and costly methods ordinarily used in direct explorations of the substrata. Where conditions warrant, the direct methods of exploration may be used on the final location selected.

Problem (*8*).—The earth-resistivity traverse or constant-depth test should be of aid to the geologist in locating buried river channels throughout a given area, particularly where sand and gravel are concentrated within such channels and where rapid erosion might conceivably occur during flood periods. The determination of the presence, depth, and lateral extent of such deposits might well influence decisions as to the desirability of a particular bridge or road location.

Problem (*9*).—Earth-resistivity tests have been used successfully in connection with explorations involving a tunnel site in a hydroelectric development in Canada (56)(57). The use of this test in explorations on highway

tunnel sites is likely to be largely influenced by the depth and character of the overburden. However, the presence of faulting and zones of shattered and highly altered or weathered rock where water may prove troublesome in tunnel construction should be rather easily determined by the resistivity test. The refraction seismic test has been used in several instances by the Public Roads Administration in surveys made in connection with the location of proposed highway tunnels. The data obtained from these surveys have not been published but are contained in the files. In such studies the broken and weathered rocks have seismic wave velocities considerably different from the characteristic velocity in the solid rock sought for the portal construction. Predictions as to the need for a lining for a particular tunnel may be made, based upon intermediate seismic wave velocities indicating a fractured, jointed rock mass through which the tunnel is to pass.

Problem (*10*).—There have been numerous instances in which the resistivity test has been used successfully in the search for underground water supplies. References (58)(59)(60) are examples of such studies. At times the water may be located more or less directly by reason of low resistivity resulting from various salts carried in solution, but more often it is located through its association with some stratigraphic or structural formation such as clean sand and gravel or a rock formation conducive to the entrapment and storage of water.

GEORGE E. EKBLAW,[16] ESQ.[16a]—The purpose of this paper (see "Synopsis") is "to help both engineers and geologists to recognize more frequently those situations in which geology does enter into highway engineering." How extended or complete the discussion should be is an open question—it could range from the extreme minimum of a single statement of obvious fact to a maximum of volumes in which all known or possible examples were cited. Whether or not the length and style of this paper is sufficient for the purpose, therefore, is largely a matter of personal opinion and hardly subject to technical discussion.

The author has chosen to subdivide the topic into twelve "problems" ("classes of problems" might be a more appropriate term). By inference these comprise the entire field of cooperative endeavor. Actually there are numerous other classes of problems which are equally or more important—such as (*a*) selecting the most desirable of two or more alternative routes for a highway, either locally or regionally, from the viewpoint of geologic conditions, (*b*) locating or alining a highway to avoid undesirable local conditions, (*c*) peat-bog problems, and (*d*) grade-fill problems—to cite from experience in Illinois alone.

Mr. Huntting has cited incidents to exemplify these "problems." Naturally these examples are those with which he is personally familiar or which have impressed him. Any one working in the dual field of geology and highway engineering could supplement these by other incidents, perhaps even more striking or typical. The number and variety of such incidents would be determined by geographic and temporal range and variety of experience, but to a greater or lesser degree Mr. Huntting's examples could be duplicated anywhere,

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[16a] Received by the Secretary April 28, 1944.

so they may be considered sufficient to illustrate the classes of engineering problems discussed.

Within the limits of the paper, Mr. Huntting has done probably as well as any one could. Because of the brevity of the paper, many generalized statements have been made which, like all generalities, may not be universally true and to which exceptions of limited application could be made. Excluding such exceptions, experience in Illinois would corroborate all the statements applicable within the state, in a general way, except one: In Illinois it has been observed that silt, not clay, is the major constituent of subgrade material most affected by frost heave, and the writer does not subscribe to the opinion (see heading, "Frost-Heave Problems") that "The clay acts as the medium through which water rises from the water table by capillarity until it reaches the frost line, where it freezes and causes an irregular upheaval of the road surface." The presence of some clay in the silt may add to the effective capillarity and to the ice-layer expansion which is the principal cause of frost heave, but material that has a large percentage of clay seems to favor neither capillarity nor ice-layer expansion. Theoretically, because of the smaller grain size of clay, capillary water should rise higher in clay than in silt, but other factors apparently operate to offset the grain size.

ALLEN S. CARY,[17] ASSOC. M. AM. SOC. C. E., AND ADE E. JASKAR,[18] JUN. AM. SOC. C. E.[18a]—Geology contributes to the solution of airport engineering problems as much as to those of highway engineering. The author has presented twelve possible problems in which a geologist's services might be profitable in highway design. Most of these have been of major importance in airport design for several years. The urgency of the military airport program, the heavy wheel loadings, and the tremendous amounts of construction materials involved have forced the designing engineers to obtain a large quantity of accurate data in a minimum length of time. In many, but not enough cases, the services of full-time geologists have been requisitioned.

In the preparation of soil profiles, in procurement of truly representative soil samples, in exploration for fill materials and aggregates, and in providing drainage data, the geologist has been like the man who charged one dollar for fixing the typewriter in a few moments. Objection elicited the statement that he charged only one cent for the fixing, but ninety-nine cents for knowing how. The geologist has been able to minimize extensive "pattern" explorations involving test pits, auger borings, etc., by his professional knowledge of probable extents of uniformity in sediments (both horizontally and vertically) and his understanding of ground-water behavior.

In addition to advising on the design of the runways, the geologist has been of inestimable aid in water supply and building foundation problems.

The author has so ably discussed the various applications of geology to highway engineering that it is not necessary to enumerate further examples. However, the relation between the engineer and the geologist does deserve

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[18] Associate Geologist, Soils Laboratory, U. S. Engr. Dept., Univ. of Washington, Seattle, Wash.
[18a] Received by the Secretary May 1, 1944.

elaboration. By far the best single publication available on this subject was written by R. F. Legget in 1939 (12).

It is satisfying to note that the constructive service of the geologist is more appreciated now than at any time in the past and bids for further recognition in the future. A review of civil engineering in the past, and its status in the present, demonstrates that the future trend is inevitable. In the earliest years of modern civil engineering, when most projects were of small or moderate scope, the engineer was able to cope with his problems by being his own geologist or by considering a large factor of safety (more correctly, factor of ignorance) in his work. When modern society began to demand larger and more elaborate structures, it became necessary for the civil engineer himself to specialize. Today a civil engineering graduate may have such a subtitle as structural, sanitary, railroad, highway, irrigation, commercial, municipal, hydraulic, or materials engineer. These special fields may even be subdivided further. Many years of general experience are required before a man can qualify for the title of civil engineer.

Upon resuming responsibility for the design of major projects, the civil engineer is so busy with general aspects that he has neither the time, nor, possibly, the training in details, necessary to obtain the geological data for design. He must either delegate this responsibility to specialists or gamble. The latter course has been taken too frequently as is demonstrated by failures in the past (12). Too often the principal use of consulting geologists is in determining the causes of failure of important engineering structures.

As stated previously, the constructive service of the geologist is becoming more appreciated, and bids for higher recognition in the future. This prophecy is supported not only by the results of mistakes that have been made in the past, but also by the fact that the most favorable sites for engineering projects are gradually being "used up." The ingenuity of the civil engineer is being taxed to the utmost by the design of large concrete and earth-fill dams, long-span bridges, large airports (often built on the poorest foundations), intricate highway systems, elaborate water-supply systems, and many other projects. Thus, the geologist is needed more than ever.

Where to find the experienced engineering geologist, who can fulfil such responsibilities, and devote full-time service to the engineer, is an important consideration, but one beyond the scope of this discussion.

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(54) **"The Earth Resistivity Method Applied to the Prediction of Materials in Excavation,"** by R. C. Schappler and F. C. Farnham, paper presented at the Twenty-Fifth Mississippi Valley Conference of State Highway Depts., Chicago, Ill., February, 1933 (unpublished).

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(58) **"The Application of Electrical Resistivity Prospecting to Ground Water Problems,"** by C. M. Tattam, *Colorado School of Mines Quarterly*, Vol. 32, No. 1, January, 1937, pp. 117–138.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

RAINBOW ARCH BRIDGE OVER NIAGARA GORGE

A SYMPOSIUM

Discussion

BY GEORGE STERN

GEORGE STERN,[58] ASSOC. M. AM. SOC. C. E.[58a]—A valuable contribution to the literature on practical design of arches of long span is contained in this Symposium. The design of an arch bridge of 950-ft span represents a structural problem of the first magnitude. The use of high-strength structural steel with high allowable unit stresses leads to structural members of small cross section and to comparatively small rigidity of the structure.

In an arch rib of long span considerable deformation will occur and design by the usual so-called "elastic theory" will involve appreciable errors, as was correctly stated by the authors of the first Symposium paper. Many of these potential errors will be avoided by the use of the approximate method suggested in the Symposium. To obtain correct results, however, the deformation of the arch under the design loading must be considered in setting up the equations of equilibrium—that is, the "deflection theory" must be used.

Although the so-called exact method, outlined in the first Symposium paper obviously can be used to determine the final moments by a trial-and-error procedure, a direct analysis when not too laborious would be preferable. The equilibrium equations required for such a direct analysis may be set up as follows:[9]

The bending moment $M(x)$ at a point x of a two-hinged arch can be expressed as

$$M(x) = m(x) - H\,[y(x) - \eta(x)] \qquad (98)$$

NOTE.—This Symposium was published in October, 1943, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1943, by Egidio O. Di Genova, and Charles Mackintosh; January, 1944, by C. M. Goodrich, and T. Kennard Thomson; February, 1944, by C. M. Spofford, L. J. Mensch, and O. H. Ammann; March, 1944, by Louis Balog, William G. Rapp, and Leon Beskin; April, 1944, by Neil Van Eenam, and C. H. Gronquist; and May, 1944, by M. Hirschthal.

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[58a] Received by the Secretary April 29, 1944.

[9] "A Deflection Theory of Two-Hinged Arch Ribs," by G. Stern, thesis No. 516, Dept. of Civ. Eng., Columbia Univ., New York, N. Y. (a résumé of theoretical studies made at the Technical Univ. of Vienna, Austria).

in which H is the true horizontal thrust of the arch and $m(x)$ is the bending moment at the same point x of a simple beam of the same span and cross section as the rib. The vertical displacement $\eta(x)$ of point x is considered positive when downward. The second derivative of Eq. 98 is

$$\frac{d^2M}{dx^2} = -W(x) - H\frac{d^2y}{dx^2} + H\frac{d^2\eta}{dx^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (99)$$

On the other hand, the following well-known relationships apply for a small element ds of a curved bar subjected to axial force, bending moment, and temperature change:

$$\frac{\Delta\, ds}{ds} = \frac{T}{E\,A} \mp \epsilon\, t \quad \ldots\ldots\ldots\ldots\ldots\ldots (100a)$$

and

$$\frac{\Delta\, d\theta}{dx} = -\frac{M}{E\,I\cos\theta} - \frac{T}{E\,A\,r\cos\theta} \pm \frac{\epsilon\, t}{r\cos\theta} \quad \ldots\ldots\ldots\ldots (100b)$$

in which $d\theta$ = the angle formed by the end tangents of the element ds, $\Delta\, ds$ = the change in length at the neutral axis, and $\Delta\, d\theta$ = the angular displacement of the element ds. The remaining notation is that used in the Symposium.

From the geometrical relations,

$$dy = ds\sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (101a)$$

and

$$dx = ds\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (101b)$$

the following expressions for the vertical and horizontal displacements, $d\eta$ and $d\rho$, are derived:

$$d\eta = \frac{\Delta\, ds}{ds}\, dy + \Delta\,\theta\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (102a)$$

and

$$d\rho = \frac{\Delta\, ds}{ds}\, dx - \Delta\,\theta\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots (102b)$$

By differentiating Eq. 102a twice,

$$\frac{d^2\eta}{dx^2} = \frac{\Delta\, ds}{ds}\frac{d^2y}{dx^2} + \frac{d(\Delta\,\theta)}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (103)$$

and, combining Eq. 103 with Eqs. 100, the following expression can be obtained:

$$\frac{d^2\eta}{dx^2} = \frac{T}{E\,A}\frac{d^2y}{dx^2} \mp \epsilon\, t\frac{d^2y}{dx^2} - \frac{M}{E\,I\cos\theta} - \frac{T}{E\,A\,r\cos\theta} \pm \frac{\epsilon\, t}{r\cos\theta} \quad \ldots\ldots (104)$$

By substituting for $T = H\sec\theta$, $\frac{d^2y}{dx^2} = -\frac{1}{r}$, introducing the notation $I\cos\theta = \bar{I}$ and $\frac{d^2\eta}{dx^2} = \eta''$, and, considering that M, as well as A and I, is a function of the abscissa x, Eq. 104 may be rewritten:

$$E\,\bar{I}(x)\,\eta''(x) = -M(x) - \frac{2\,H}{r}\frac{I(x)}{A(x)} \pm \frac{2\,E\,\bar{I}(x)}{r}\epsilon\, t \quad \ldots\ldots\ldots (105)$$

In general, the second term of the right side of Eq. 105, representing the effect

of the normal stresses due to the axial thrust, has small influence on the results. The second derivative of this equation, substituted into Eq. 99, gives

$$[E\,\bar{I}(x)\,\eta''(x)]'' + H\,\eta''(x) = +\,W(x) + H\,y''(x) \ldots\ldots (106)$$

Eq. 106 expresses the vertical displacement of a two-hinged arch rib as a function of the horizontal thrust H. In differentiating Eq. 105, the second term dependent on H was neglected. In the latter part of this discussion, however, the final solutions will also be given for the case in which that term is not disregarded.

Another relationship between horizontal thrust and vertical displacement can be derived from the fact that the relative displacement of the two hinged supports must be zero. Hence, from Eqs. 102, after some transformation,

$$\int_0^L d\rho = \int_0^L \eta'\,y'\,dx - \int_0^L \frac{\Delta\,ds}{ds}\,[1 + (y')^2]\,dx = 0 \ldots\ldots (107)$$

By substituting Eq. 100*a* and introducing an arbitrary cross-sectional area A_o, Eq. 107 can be transformed mathematically into

$$\int_0^L \eta(x)\,y''(x)\,dx + \frac{H}{E\,A_o}\int_0^L \left\{ \frac{A_o}{A(x)\cos\theta_x}\,[1 + (y')^2] \right\} dx$$
$$\mp\,\epsilon\,t \int_0^L [1 + (y')^2]\,dx = 0 \ldots\ldots\ldots\ldots\ldots (108)$$

Eqs. 106 and 108 represent two formulas with two unknowns, the horizontal thrust H and the vertical displacement η. As soon as the general solution of the differential Eq. 106 is obtained, H and η can be computed.

Eq. 106 is a nonhomogeneous differential equation of the fourth order; the derivation of its general solutions is far too laborious to be given in detail. The procedure followed was first to solve the corresponding homogeneous differential equation. In the case of an arch rib of constant cross section (that is, $\bar{I}(x) = I(x)\cos\theta = \text{constant}$), the characteristic values λ_i and characteristic functions $\phi_i(x)$ of Eq. 106 can easily be computed since they represent simple sine curves. Thus the vertical displacement and the moment can be expressed in the form of Fourier series. On the other hand, if the moment of inertia of the rib varies, the characteristic functions are no longer true sine curves. To obtain these $\phi_i(x)$-values, some additional preparatory computation must be made. The formulas that can be derived to express these sinusoidal curves are not too elaborate, but their complexity increases the more closely the variation of the moment of inertia is considered.

Once the characteristic values λ_i and functions $\phi_i(x)$ of the corresponding homogeneous differential equation are computed, the general solution of the nonhomogeneous differential equation can be expressed by a uniform and convergent infinite series of these previously mentioned functions. This method of solving by series was first used by H. H. Bleich in his deflection theory of suspension bridges.[59] Just as the theory of arches shows a good analogy to

[59] "Die Berechnung verankerter Hängebrücken," by Hans H. Bleich, Springer, Vienna, 1935.

that of suspension bridges, so also do the respective deflection theories correspond rather well if similar methods of attack are used. In the case of two hinged arch ribs, the aforementioned series converges so rapidly that, for all practical purposes, it is more than satisfactory to use only a few terms of the series. Thus, the general solution of Eq. 106 may be written

$$\eta(x) = \sum_{i=1}^{i=\infty} d_i \, \phi_i(x) \ldots\ldots\ldots\ldots\ldots\ldots (109)$$

in which the factor d_i is given by

$$d_i = \frac{a_i + b_i H}{\lambda_i - H} \ldots\ldots\ldots\ldots\ldots\ldots (110)$$

The parameters a_i and b_i are defined by the integrals,

$$a_i = \int_0^L w(x) \, \phi_i(x) \, dx \ldots\ldots\ldots\ldots\ldots\ldots (111a)$$

and

$$b_i = \int_0^L y''(x) \, \phi_i(x) \, dx \ldots\ldots\ldots\ldots\ldots\ldots (111b)$$

As stated previously, Eq. 109 can now be substituted in Eq. 108. Thus, by integrating term by term and by transformation, an expression can finally be derived which contains the horizontal thrust H as the only unknown value:

$$\sum_{i=1}^{i=\infty} \frac{(a_i + b_i H) \, b_i}{\lambda_i - H} + \frac{C}{E \, A_o} H \mp C_t \, \epsilon \, t = 0 \ldots\ldots\ldots\ldots (112)$$

in which

$$C = \int_0^L \frac{A_o}{A(x) \cos \theta_x} [1 + (y')^2] \, dx \ldots\ldots\ldots\ldots (113a)$$

and

$$C_t = \int_0^L [1 + (y')^2] \, dx \ldots\ldots\ldots\ldots\ldots\ldots (113b)$$

In Eqs. 109 and 112, the effect of the normal stresses due to the thrust is disregarded. It was mentioned before that this effect on the vertical displacement is very small. Thus, in most cases, Eqs. 109 and 112 will give satisfactory results.

However, if the effect of the normal stresses is not neglected, the general solution of the differential equation of the two-hinged arch problem may be expressed by

$$\eta(x) = \sum_{i=1}^{i=\infty} \frac{a_i + b_i H}{\lambda_i - H} \phi_i(x) - \frac{2 \, I_A}{r \, A_A} \left[1 - \frac{\cos \alpha \left(x - \frac{L}{2} \right)}{\cos \frac{\alpha \, L}{2}} \right] \pm \frac{2 \, E \, I_A \cos \theta_A}{r \, H}$$

$$\left[1 - \frac{\cos \alpha \left(x - \frac{L}{2} \right)}{\cos \frac{\alpha \, L}{2}} \right] \epsilon \, t \ldots\ldots\ldots\ldots\ldots\ldots (114)$$

in which $\alpha = \sqrt{\frac{H}{E\,I_A \cos\theta_A}}$. In the terms representing the effect of the normal stresses, average values A_A, I_A, and θ_A are used for area, moment of inertia, and the angle θ. Using Eq. 114, the following final formula can be developed:

$$\sum_{i=1}^{i=\infty} \frac{(a_i + b_i H)\, b_i}{\lambda_i - H} + \frac{C}{E\,A_o} H + \frac{2\,I_A}{r^2\,A_A}\left(L - \frac{2\tan\frac{\alpha L}{2}}{\alpha}\right)$$

$$\mp C_t\,\epsilon\,t \mp \frac{2\,E\,I_A\cos\theta_A}{r^2\,H}\left(L - \frac{2\tan\frac{\alpha L}{2}}{\alpha}\right)\epsilon\,t = 0 \ldots\ldots\ldots (115)$$

The derivation of Eqs. 109, 112, 114, and 115 shows that they can be applied to any symmetrical two-hinged arch, irrespective of the shape of the arch axis, the variation in cross section of the rib, and the loading condition to be investigated.

The horizontal thrust H can now be calculated from Eq. 112 or 115 by the trial-and-error method. Once its value is known, the bending moment, shear, and unit stress may easily be computed by the conventional methods, and the vertical displacement can immediately be obtained from Eq. 109 or 114. (The writer is informed that Friedrich Bleich, M. Am. Soc. C. E., has developed, but not yet published, a deflection theory of two-hinged and hingeless arches. He uses a method similar to that outlined herein, but, by neglecting the effect of the normal stresses due to the thrust on the vertical displacement, equations are derived from which H can be calculated directly.)

A solution for the final moments in the preliminary two-hinged Niagara arch gave results which agree with those obtained by the authors of the first Symposium paper. Using the method outlined herein, the writer computed a 60% increase in the moment at the quarter point due to the deformation of the arch rib, as compared with the 63% increase obtained by the so-called exact procedure in the first Symposium paper.

Although few engineers will have the opportunity to design an arch bridge of the magnitude of the Rainbow Arch Bridge, the record of its design and construction will prove very valuable to future designers of similar projects.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

HYDRAULIC MODEL INVESTIGATION OF MITER GATE OPERATION

Discussion

BY EDWARD SOUCEK

EDWARD SOUCEK,[3] ASSOC. M. AM. SOC. C. E.[3a]—Prior to adoption of the test program so ably described by Mr. Amster, a study was made of the methods which had been used previously in design of miter gate moving machines. The usual basis for the hydraulic load is a differential head on the leaves acting in a direction opposing the motion. The magnitude of the head is commonly fixed by judgment—in some cases supplemented by observations at existing installations. The reliability of comparative studies for use in design is limited principally by uncertainty in allowing for differences in gate length, in submergence depth, and in kinematics of the gate motion; sill clearances are also believed to affect the hydraulic loads but probably to a lesser extent.

Some designers compute the varying differential head upon the leaves by consideration of the rates of change of the areas separated by the gates and the discharge past the gates. This method gives results that are much too small because the loads are principally due to water-level variations which are transient in character and local in extent—surges induced by the gate movement —and therefore not related in any simple manner to the adjacent water-surface areas.

During the search for design data, a paper that dealt with loads on moving miter gates was found (written in German). It appeared very promising because it contained many equations. Upon translation, however, all significant equations were found to contain that elusive term—the unknown and varying differential head—which had to be evaluated before the equations could be used in design. Owing to the lack of reliable design data, use of model tests was, as the author states, the only alternative to the use of extremely liberal safety factors for the design of a considerable number of large machines.

NOTE.—This paper by Maurice N. Amster, Assoc. M. Am. Soc. C. E., was published in March, 1944, *Proceedings*.

[3] Senior Civ. Engr., U. S. Engr. Office, Wilmington, N. C.

[3a] Received by the Secretary May 22, 1944.

This situation not only gives Mr. Amster's paper a "scarcity value" which insures that it will be widely used, but it also requires that most careful consideration be given to the limitations stated by the author and to any other conditions that might affect the applicability of the test results to the design of gate moving machines in general. Operation of the model and interpretation of the test data in accordance with Froude's law are believed correct since the phenomena associated with the hydraulic loads are predominantly inertial in character. Elasticity of the gates, effect of flow through the lock culverts, and reduction in effective chamber length caused by the nearness of a large ship might be mentioned as factors not represented in the model additional to those listed by the author. Their effects are relatively small but should not always be considered negligible.

Under "Application of Test Results to Other Miter Gates," the author outlines a method for application of the test data to the design of dissimilar prototypes. The writer believes that the problem is not so simple as this section of the paper implies. If gate moving machines are not geometrically similar, the gate operating time is not a sufficient measure of the kinematic characteristics of the gate movement. If the model gates were operated at a uniform angular velocity, rather than with slow movement near the limits of travel, the hydraulic load for the same operating time would be increased many times. For this reason, it would be preferable to consider only the first 15 or 20 degrees of the movement for the critical opening operation rather than the entire opening period in some cases.

However, even dismissing the questions of kinematic characteristics (other than operating time), leaf angle when fully open, sill clearance, and other factors which were not investigated, it is not clear that the proposed adjustment for operating time is generally valid. As the author explains, the ratio of prototype leaf length to model leaf length establishes the scale ratio. Then from Fig. 6(*b*), a torque curve can be selected that is correct for a particular prototype operating time (determinable from the scale ratio and the model operating time). It may, but probably does not, correspond to that at which the prototype will operate. Fig. 6(*a*) shows the effect of varied operating time only for a gate with leaf length approximately equal to the submerged depth. The effect of operating time upon peak torque indicated by the curves is shown in Table 2. It does not follow from these data that Fig. 6(*a*) is a valid basis

TABLE 2.—EFFECT OF OPERATING TIME ON PEAK TORQUE

Description	TIME, IN MINUTES				
	1.75	2.00	2.25	2.50	2.75
Torque, in millions of foot-pounds	22.3	20.0	16.5	14.5	12.5
Percentage time increase	0.0	14.4	28.5	43.0	57.5
Percentage torque decrease	0.0	10.5	26.0	35.0	44.0

for adjustment of torque curves in terms of operating time when the leaf length is much greater than the submerged depth—a frequent case in barge locks. Pending the presentation of further data or a conclusive analysis, it

would appear that the relationships suggested should be used only when the leaf length approximately equals the submerged depth.

The treatment of the forces due to density differences covers the ideal case, a static condition with the plane separating fresh and salt water in a known position. In reality, determination of the elevation of this plane is quite difficult. It may be of interest to point out that a series of tests on the existing Panama locks, corroborated by a small transparent model, indicated that the excess closing force produced by difference in density cannot be eliminated by increasing the elevation of the culvert outlets. Consideration of this problem in detail is beyond the scope of this discussion. It may be suggested, however, that designers of miter gate moving machines should question the efficacy of elevated outlets in reducing the loads upon the machines.

In the Panama Canal Third Locks Project, the terminal gates which are most subject to the density effect will be less deeply submerged than most of the other gates so the machines of the terminal gates will have a substantial overload capacity. At all gates, the "overtravel" of water in the lock culverts produces a transient opening force which can be used to reduce the critical opening load, either by swinging the gates at or near the time of maximum overtravel or by closing the culvert valves and "trapping" the favorable water level differential. Of course a falling tide eliminates the latter possibility at the terminal gates.

The writer's remarks are intended solely to clarify certain design problems and to emphasize the limitations upon general applicability of the test data. Many significant conclusions can be drawn from the paper in addition to those modestly claimed by the author. His efforts would be justified if the paper did no more than call attention to the large reduction in operating forces and peak power loads which can be attained by reducing the operating speed of the gates and by nonsynchronous operation of the leaves. Since the gate operating time is a relatively small part of the time required to perform a lockage, these methods frequently can be adopted with scarcely noticeable adverse effects upon over-all lockage efficiency.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

STORAGE AND THE UNIT HYDROGRAPH

Discussion

BY FRANKLIN F. SNYDER

FRANKLIN F. SNYDER,[26] ASSOC. M. AM. SOC. C. E.[26a]—Papers dealing with unit hydrograph and flood routing procedures have been quite numerous in recent years and the author is to be complimented for his efforts to clarify the inherent relationship between these hydrologic tools. However, the extent to which that objective has been accomplished is modified by the introduction of a number of confusing and unsupported statements about certain hydrologic phenomena.

In the "Introduction," the author, referring to an article,[7] states that "For example, in determining a seven-period unitgraph from a flood resulting from three periods of runoff, it is possible to determine thirty-six unit hydrographs which, applied to the three items of runoff, will reproduce the composite hydrograph exactly." Theoretically, if the unit hydrograph procedure were exact, there would be one unit hydrograph that would reproduce the composite hydrograph exactly. For illustration, one can assume a unit hydrograph and compute a composite hydrograph. There exists, therefore, a unit hydrograph which will reproduce the composite hydrograph exactly. In actual practice there is no unit hydrograph that will reproduce observed composite hydrographs exactly. However, if the composite hydrograph is not the result of a too complex storm, numerous methods are available for determining a unit hydrograph that will reproduce the observed composite hydrograph with a satisfactory degree of accuracy. The example[7] cited by the author illustrates this theory, as the most probable distribution-graph values obtained by "least squares" for a composite storm do not reproduce exactly the observed hydrograph.

Fig. 3 shows the results of a study of the effect of storage that is equally a function of inflow and outflow on a wave form with an initially very sharp

NOTE.—This paper by C. O. Clark was published in November, 1943, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1944, by James S. Sweet, and Otto H. Meyer; March, 1944, by L. K. Sherman, Gordon R. Williams, and Ray K. Linsley, Jr.; and May, 1944, by E. F. Brater, L. C. Crawford, Robert E. Kennedy, and Victor H. Cochrane.

[26] Hydr. Engr., War Dept., Office of Chf. of Engrs., Washington, D. C.

[26a] Received by the Secretary May 15, 1944.

[7] "Engineering Procedure as Applied to Flood Control Reservoirs with Reference to the Muskingum Flood Control Project, Appendix IV," The Engineer School, Fort Belvoir, Va., 1936.

peak. The author has made a common mistake[27] in obtaining an outflow hydrograph different from the inflow hydrograph after assuming that the reach storage is equally a function of inflow and outflow ($x = 0.5$, or that storage varies as the sum of inflow plus outflow).

The fact that the outflow hydrograph should be identical with the inflow hydrograph except for a time shift is readily proved from the general storage equation and is evident from a consideration of the Muskingum formulas, Eqs. 6*d*, 7*a*, 7*b*, and 7*c*. Another result is obtained when certain basic requirements fundamental to most routing procedures are violated:[8] (1) The routing interval, T, must be small enough to define, satisfactorily, variations in rate of flow; and (2) the reach should be of such length that the time of travel is approximately equal to T. In the example under discussion a 2-hr value of time of travel, K, was assumed by the author for an imaginary reach of a river. The value of T that was used to derive the hydrographs in Fig. 3 is not given, but, according to requirement (2), T should be 2 hours. Then, with x equal to 0.5, $C_0 = 0$, $C_1 = 1.0$, $C_2 = 0$, and $O_2 = I_1$—that is, outflow at the end of the period equals inflow at the beginning. Requirement (1) was violated since a value of T equal to 2 hours is too large to define, adequately, the hydrograph in Fig. 3. However, O_2 will always equal I_1 from Eq. 6*d* as long as T is made equal to K.

On the same basis the outflow curve labeled $x = 0.5$ in Fig. 2 is in error and should be identical to the original curve except for a time shift of 4 hours. The negative values for the curve, $x = 0.5$, and for the other outflow curves, also, are definitely due to the selection of units and the limitations of the routing procedure, and to couple these values with physical phenomena either real or imaginary seems unwise.

Another erroneous result is often obtained when routing a hydrograph such as that for Danville in Fig. 12, through a reach with a value of K considerably greater than a reasonable value of T. The slow prolonged rise and rapid fall coupled with the assumptions of the Muskingum routing procedure result in a peak outflow rate higher than the inflow rate even though a value of x considerably less than 0.5 is assumed. Walter B. Langbein,[28] Assoc. M. Am. Soc. C. E., has suggested an adjustment of the Muskingum procedure to compensate for the inability of the weighted flow to represent storage in the reach adequately when either inflow or outflow is changing rapidly.

To justify the basic assumptions followed in developing the Muskingum routing procedure, the value of T should be made approximately equal to the value of K. If this is not done (and it is seldom possible in practical applications), the value of x is a function of the length of reach, or K, in addition to indicating the portion of the weighted flow that is derived from the inflow.

The attempt to prove that shifting a hydrograph in time and routing it through a small value of storage gives a hydrograph quite similar to that

[27] "A Direct Method of Flood Routing," by C. O. Wisler and E. F. Brater, *Transactions*, Am. Soc. C. E., Vol. 107 (1942), p. 1519.

[8] "Flood Routing," by Edward J. Rutter, Quintin B. Graves, and Franklin F. Snyder, *ibid.*, Vol. 104 (1939), p. 278.

[28] "Channel Storage and Unit Hydrograph Studies," by Walter B. Langbein, *Transactions*, Am. Geophysical Union, Pt. II, 1940, pp. 620–627.

obtained with the Muskingum routing procedure using a value of $x = 0.5$ has resulted in a failure to make clear an important objective of the basic idea. Most flood routing procedures actually provide no time advantage for forecasting purposes—that is, the inflow data must be extended into the future just as far as it is desired to obtain values of outflow. The procedure proposed by the author—namely, shifting the inflow hydrograph in time and routing it through a predetermined amount of reservoir-type storage—does give a time advantage equal to the amount of the time shift.

The writer has been accumulating data in connection with such a routing assumption and, as would be expected from the ease with which routing procedures can be verified, has had little difficulty in obtaining satisfactory results when routing observed hydrographs for various streams in the eastern United States.

In analyzing these records a procedure convenient for routing the transposed hydrograph through a certain amount of reservoir-type storage was developed. As stated by the author, where storage, S, and outflow, O, are uniquely defined by the elevation at the outlet such as in a reservoir ($x = 0.0$ in the Muskingum procedure), the ratio of successive increments of storage to increments of discharge, $\frac{\Delta S}{\Delta O}$, has the dimension of time and is represented in this discussion by the symbol T_s. If storage is in acre-feet and O in cubic feet per second, T_s is in units of days and equals (very nearly) $0.5 \frac{\Delta S}{\Delta O}$. Starting with a volumetric expression of the storage equation for some satisfactory routing period, the mean inflow, $\bar{I}$, minus the mean oufltow, $\bar{O}$, equals the change in storage. For units of cubic feet per second for discharge, acre-feet for storage, and days for time:

$$\bar{I} - \bar{O} = \frac{\Delta S}{2t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

With the usual assumption of straight-line variation of inflow and outflow during the routing period, t, and, since $\bar{O}$ then equals $\frac{\Delta O}{2} + O_1$, with the subscript 1 indicating the value of the variable at the beginning of a routing period:

$$\bar{I} - O_1 = \frac{\Delta O}{2} + \frac{\Delta S}{2t} = \Delta O \left(0.5 + 0.5 \frac{\Delta S}{t \Delta O}\right) \quad \ldots\ldots\ldots\ldots (20)$$

Substituting T_s for $0.5 \frac{\Delta S}{\Delta O}$ and solving for ΔO gives

$$\Delta O = C (\bar{I} - O_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

in which

$$C = \frac{1}{0.5 + \frac{T_s}{t}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

This solution of the storage equation lends itself readily to tabular computation and to the use of variable values of T_s by plotting the variation of C

against O. When inflow is zero, ΔO equals $-C O_1$ and C equals $1 - C_r$ in which C_r is a recession coefficient of the outflow hydrograph equal to $\frac{O_2}{O_1}$.

Fig. 17 shows graphically some of the time relationships utilized in analyzing observed records. Local inflow has been separated from the outflow hydrograph so that the latter represents the total effect of the transposition, T_t, and the reservoir-type storage, T_s, on the hydrograph of inflow at the upper

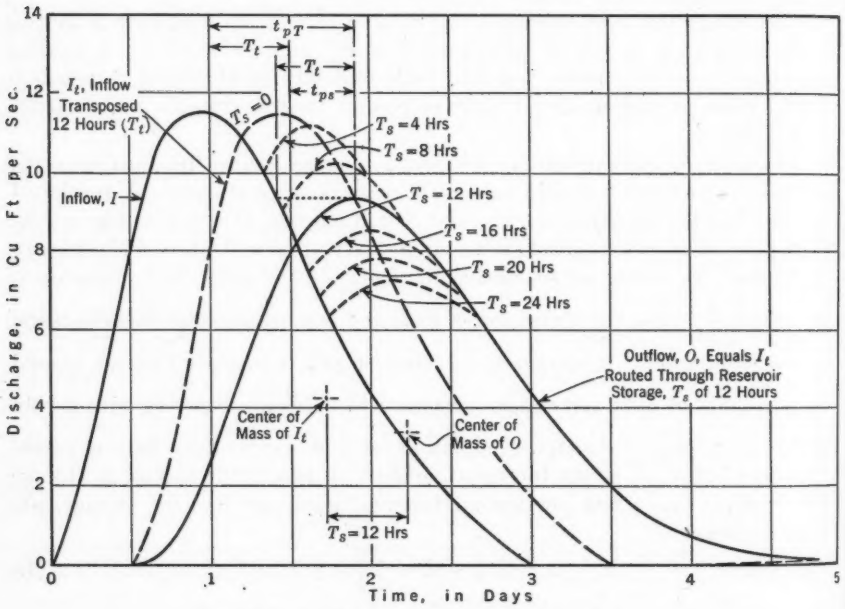

FIG. 17.—TIME AND STORAGE RELATIONSHIPS

end of the reach. The total time shift between center of mass of inflow and outflow, T_T, is equal to the sum of T_t and T_s; that is,

$$T_T = T_t + T_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

The value of T_T is also equal to the value of K in the Muskingum routing procedure provided the fundamental routing requirements previously discussed are not violated. However, if the inflow hydrograph, I, of Fig. 17 is routed using the Muskingum coefficients with $K = 24$ hours, $x = 0.40$, and $t = 6$ hours (too long a reach), the time lag between center of mass of inflow and outflow, T_m, will be 21 hours instead of 24 hours. A quite similar outflow hydrograph is obtained by this procedure, using $T_t = 10$, $T_s = 11$, and $T_T = 21$ ($T_m = 21$).

The value of T_t in any particular case is available from inspection of the hydrograph and the following relationship is useful in estimating values of T_t: From Fig. 17, t_{pT} equals T_t plus t_{ps}, in which t_{pT} is the total time lag of the

crest discharge, T_t is the crest lag due to the transposition, and t_{ps} is the lag of the crest discharge due to reservoir-type storage effect; therefore:

$$t_{pT} = T_t + t_{ps} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

and, since T_s equals $T_T - T_t$,

$$T_s = T_T - t_{pT} + t_{ps} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The time between center of mass of the inflow and outflow hydrographs, T_T or T_m, is readily obtained by transferring the hydrograph shapes with carbon paper to a sheet of stiff paper, cutting out the hydrographs, and locating the centers of area.

In the beginning the writer had hoped that values of T_t and T_s could be assumed constant throughout a wide range of flow conditions, an assumption that is theoretically incorrect but one that is made in most routing procedures. A variation of T_s can be handled without much difficulty but when both T_t and T_s vary so that their sum also varies, the problem becomes more difficult. As might be expected, observed hydrographs for many streams could be duplicated by assuming constant values of the time and storage factors. However, when a stream with a wide flood plain such as the Missouri River goes overbank, there is a variation of several hundred per cent in the factors.

Under the heading, "Valley Storage," the author states that "The reservoir pool above a dam may not reduce flood discharges." Most engineers will subscribe readily to this generality, and its acceptance is indicated by the fact that considerable attention is given to this problem in the design and operation of reservoirs for hydroelectric projects. The author also states that, "* * * if the pool level at the dam is held constant by gate control, * * * the peak outflow may be larger than the peak inflow." The writer could also agree with this statement if the words "natural peak" discharge at the dam site were substituted for the words "peak inflow"; but it will take considerable more evidence than the Muskingum routing hypothesis to convince him that the peak outflow can be greater than the peak inflow. As a matter of fact, because of backwater storage above a level reservoir pool, the level of the pool at the dam would have to be lowered somewhat to make the peak outflow of a flood as large as the peak inflow. The peak inflow in some cases is larger than the natural peak at the dam site would have been prior to construction of the project because of the elimination of some valley storage but principally because of the change in timing of the concentration of the runoff from the areas draining directly into the reservoir.

The author's extension of storage theory to ground-water flow is quite interesting. The use of a unit hydrograph for distributing ground-water runoff may be quite practicable, but the use of storage values determined from recession curves for deriving the rising part of the runoff hydrograph is subject to more criticism in the case of ground-water runoff than in the case of surface runoff, and the procedure is open to question even for surface runoff.

Other than the use of a constant factor for dividing total runoff into surface and ground-water runoff, the author's procedure probably would be satisfactory

for many purposes. However, the writer takes exception to the inference that, in comparison, current practice in treating ground-water discharge is unsatisfactory. Reference is made to a paper[29] discussing this and other phases of runoff phenomena in which both direct runoff and ground-water runoff for the Schuylkill River above Pottstown, Pa. (drainage area, 1,147 sq miles) were computed storm by storm for a year of record using only observed values of precipitation and temperature. This article presents a graphical comparison of the computed ground-water discharge and the observed hydrograph.[30] The accuracy of the ground-water phase of the computations proved more than sufficient for any need of the writer in flood forecasting or hydrologic studies.

The author's use of the quantity, K, and the time of concentration in the development of unit hydrographs is confusing. Under the heading, "The Hydrograph as an Index of Storage," it is stated that, at the time of most rapid relative rate of decrease of discharge, "The falling legs of a large number of hydrographs of flood discharge for streams in major drainage basins from New Hampshire to North Carolina exhibit a ratio of decrease of discharge to total discharge which, for a value of $x = 0.5$, indicates a value of K closely approximating the time elapsing between termination of runoff producing rainfall and the point of most rapid relative decrease in discharge." With $x = 0.0$ the values of K for these same recessions determined by Eq. 12*b* would be equal to one half the value of time of concentration ($0.5\ T_c$). In the "Summary" it is stated that the time-area concentration curve should be routed through reservoir storage "equal to K times the outflow when K is the time indicated by Eq. 12*b*" at the time of most rapid relative rate of decrease of discharge when $x = 0.0$; that is, $K = 0.5\ T_c$. However, the author states (see heading, "Determination of the Instantaneous Hydrograph for Streams Without Flow Records") "the amount of storage by which the time-area concentration curve should be modified will approximate 10% to 25% of the time of concentration * * *." In the example for the Appomattox River above Petersburg, a value of $K = 0.06\ T_c$ was used ($K = 9$ hours, $T_c = 144$ hours) whereas the average value of K for the James River at Lick Run as determined from the data in Fig. 5 was $0.5\ T_c$ ($K = 14$ hours, $T_c = 28$ hours). With a somewhat similar procedure, which is about the most common and oldest known rational method of determining synthetic hydrographs, the writer has obtained satisfactory unit hydrographs from time-area curves routed through storage values equal to the unit hydrograph "lag" defined as time from center of mass of effective rainfall to peak of the unit hydrograph. Values of "lag" are almost always greater than one half the time of concentration as defined by the author.

Howard M. Turner, M. Am. Soc. C. E., and Allen J. Burdoin,[31] using a similar procedure, obtained fairly good reproduction of observed hydrographs for the Connecticut River at Sunderland, Mass., by routing a time-area or

[29] "A Conception of Runoff-Phenomena," by Franklin F. Snyder, *Transactions*, Am. Geophysical Union, Pt. IV, 1939, pp. 725–738.

[30] *Ibid.*, Fig. 4, p. 728.

[31] "The Flood Hydrograph," by Howard M. Turner and Allen J. Burdoin, *Journal*, Boston Soc. of Civ. Engrs., Vol. XXVIII, No. 3, July, 1941.

inflow hydrograph through a reservoir storage for which K equals 59 hours with a value of time of concentration equal to 50 hours ($K = 1.2\ T_c$).

The principal purpose of the preceding discussion is to emphasize the fact that there are still two variables in the development of a synthetic unit hydrograph: A time element (time of concentration, T_c, in this paper), and a storage factor (expressed as a percentage of T_c by the author). The use of a slope factor has been introduced by the author as an aid in determining the time factor, but the storage factor has not been correlated directly with any physical features of the drainage basin.

From the writer's experience in attempting to correlate the two variable factors with physical features,[32] he is not too optimistic about the author's use of a slope factor to determine the time variable except in localized areas. The river slope adopted by the author is believed to be a more representative slope factor than the average over-all slope usually obtained by dividing the total fall by the length. However, the data presented in the paper indicate a range of values of C, in which $T_c = C \frac{L}{\sqrt{s}}$, from 0.6 for the Smith River at Bassetts to 2.0 for the Appomattox River near Petersburg. (A value of average slope, as defined by the author, of 11.6 ft per mile for a length of 37 miles was determined from available reports and used for the Smith River at Bassetts instead of the value of 16.6 given in the paper.) Although the writer believes the exponent of the slope factor should be smaller than 0.5, variation of the exponent of s in the expression $T_c = C \frac{L}{\sqrt{s}}$ would have little effect on the relative range of values of C which vary from 0.6 to 2.0 for the Virginia drainage basins discussed in the paper.

The suggestion of the author that the determination of the time and storage factors for streams without records should be based on actual measurements should be emphasized. A time factor such as "lag" [32] can be obtained for any river station where discharge is principally a function of stage, without installing any equipment, by observing the time of crest stage and comparing it with field observations or official records of the rainfall characteristics of the storm. A storage factor is not so easily determined but it is not subject to as great a range in value as the time factor and the time factor provides a partial index of the actual value of the storage factor.

[32] "Synthetic Unit-Graphs," by Franklin F. Snyder, *Transactions*, Am. Geophysical Union, Pt. I, 1938, pp. 447–454.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SURGES IN PANAMA CANAL REPRODUCED IN MODEL

Discussion

BY C. J. POSEY

C. J. POSEY,[8] ASSOC. M. AM. SOC. C. E.[8a]—Prototype-model comparisons of this type are all too infrequent in engineering literature. When the underlying theory is generally accepted, there is a tendency to adjust the design to fit the results of the model tests and to omit subsequent tests on the prototype. Only when the model theory is the subject of controversy is the laborious and difficult job of securing prototype data for comparisons likely to be undertaken. Many hydraulic engineers would consider satisfactory model-prototype correspondence, in this particular problem, to be almost a foregone conclusion and prototype tests unnecessary. The writer disagrees with this viewpoint, and believes that such comparisons should be undertaken whenever practicable.

Although substantial agreement with respect to wave velocities is evident from inspection of Figs. 3 and 4, a statistical study of comparative water-surface elevations might give a more definite basis for the authors' conclusion (see heading, "Comparison of Model and Prototype") that "The agreement is considered to be very satisfactory." Plotting of the model and prototype test data in the form shown in Fig. 2 should make it easier to judge the degree of correspondence.

The authors' comparison of the results of synchronization and of anti-synchronization, shown in Fig. 4, certainly is of special interest. It seems safe to assume that these are the extreme cases and that surges resulting from operation at intermediate periods will not exceed the values obtained for synchronized operation.

NOTE.—This paper by F. W. Edwards and Edward Soucek was published in January, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1944, by Harold A. Weggel; and May, 1944, by Harry Leypoldt.

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[8a] Received by the Secretary May 29, 1944.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MILITARY AIRFIELDS
A SYMPOSIUM

Discussion

BY CLARENCE S. JARVIS, D. D. MCGUIRE, E. D. PARMER, A. CASAGRANDE, AND HARRY E. COTTON

CLARENCE S. JARVIS,[15] M. AM. SOC. C. E.[15a]—Of untold value is the record in this Symposium on the logical development of both construction and design problems, together with the solutions permitted under the strict limitations imposed by the expanding, ever-changing, and accelerating activities of wartime. Colonel Stratton's interesting notes relating to subgrade support and the various methods of stabilization by compaction, or by the addition of base courses suitably rolled, recalled some of the writer's early construction experience. In addition to the usual provisions for both surface and subdrainage, it was sometimes necessary to insure against both lateral and upward flow of compressible and plastic materials. One of the most satisfactory methods was the placement of crushed slag or stone in trenches for compaction under pile-hammer impact. The virtual interlocking of the angular fragments accomplished in this manner seemed to make such footings react very similarly to plain concrete slabs. Unquestionably the inertia of such heavy masses among the footings accounts in part for their remarkable stability under varying applications of live load and required dead load.

The writer vividly recalls the announcement in the division of design of a federal bureau concerning proposed thickened edges of pavement strips to replace either the prevailing uniform thickness, or even the heavily crowned road pavement, with edges of concrete thinned by the amount of rise in the crown. No less startling was the introduction of dowels to extend across all construction and expansion joints, thus preventing the uneven settlement that had proved so destructive to concrete under heavy metal tires.

NOTE.—This Symposium was published in January, 1944, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: February, 1944, by James B. Newman, Jr.; March, 1944, by Thomas E. Stanton; April, 1944, by W. E. Howland, and David S. Jenkins; and May, 1944, by William E. Rudolph, Raymond L. Irwin, G. G. Greulich, Hibbert Hill, Jacob Feld, and Robert Horonjeff.

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[15a] Received by the Secretary April 10, 1944.

Colonel Stratton's frequent references to traffic distribution, repeated stresses, and fatigue of pavement materials indicate his familiarity with, and his practical application of, the basic principles disclosed by the experiments of the late J. L. Van Ornum, M. Am. Soc. C. E. (37),[15b] and others. How else could any one justify the departures from previous practices described under the heading, "Concrete Paving: Allowable Working Stresses for Concrete Pavements"? The use of a safety factor of 1.75, corresponding to 57% of the 28-day flexural strength of the pavement, or even the use, for limited operations, of a safety factor of only 1.40, representing 71% of the stress ordinarily required to cause failure with a single application of loading to destruction, is entirely logical in view of the increasing strength after 28 days and the wide dispersion of traffic, as well as the wing lift accompanying the landing of planes. Who could have foreseen that the vibratory strains of the taxiways and the warming-up aprons, rather than the stresses due to landing impact, would govern both current and future design?

Summarizing the conclusions that may be drawn from the Van Ornum series of tests, application of 50% or less of the loading that would normally cause failure may be repeated indefinitely without appreciably decreasing the resistance of reinforced concrete columns, beams, or slabs; but, for a 75% loading or greater, only a few such applications should be tolerated because of the definite tendency toward fatigue of materials, accompanied by a measurable reduction of ultimate strength.

The observed variation in behavior among individual test specimens employed by the writer in a research conducted in 1905 and 1906 (38), and also among the more numerous samples subjected to the more elaborate tests of the Van Ornum series, would occasionally becloud the results. For example, a loading of 85% would cause failure with a single application, and the next specimen would withstand a similar loading 100 or more times.

In the light of all available tests to date, it is difficult to improve on the range indicated in Colonel Stratton's treatment of allowable working stresses—57% of the ordinary 28-day strength of resistance, or for limited operations, 71%.

Although there are valid objections to the use of concrete pavements as thick as 10 to 14 in., because of the attendant temperature stresses and also because of the difficulty in securing the required compaction and density, there appears to be no practicable limit on the base-course thickness of stone or slag, duly compacted. In the writer's experience, it has been demonstrated repeatedly that a base course of rubble or of quarry stone, with interstices mostly filled with fragments, would support a lightly reinforced pavement of 3-in. thickness plus whatever penetration was attained by keying into the stone base, with better results and at less cost than would plain concrete of double or treble the pavement thickness, and without base course.

Mr. Hathaway deals with noteworthy accomplishments in a broad field that seems to have awaited a well-organized, vigorous, clear-sighted, and analytic approach. Some other attempts have fallen far short of the goal because

[15b] Numerals in parentheses, thus: (37), refer to corresponding items in the Bibliography at the end of the Symposium, and at the end of discussion in this issue.

of the lack of required basic data. However, additional material has been supplied in recent years through the cooperative efforts of various federal departmental representatives including, notably, Mr. Hathaway and Merrill Bernard and N. C. Grover, Members, Am. Soc. C. E., their administrative supervisors and technical staffs, legislative committees, and, finally, the great mass of enlightened citizenry who favored liberal support of hydrometeorological and related research. In addition to this financial and scientific support, the development of fundamental procedures as exemplified by the Symposium certainly required the wholehearted assistance of specially qualified members of the technical staffs in various field offices under the Chief of Engineers, as well as of those men immediately under Mr. Hathaway's direction, as tacitly implied or expressed under "Acknowledgments."

If attempts were made to include in the "Bibliography" such technical papers and other publications as have contributed vitally to the background and to the fundamental knowledge of airfield engineering (including meteorological, infiltration, runoff, and related phenomena), the list would be too long and cumbersome for practical purposes.

The writer realizes that any constructive suggestions for either amplification or clarification of concepts may be mistaken for criticism, no matter how helpfully intended. However, passengers on maritime vessels or on other transportation units, who fail to communicate vital information that may be beyond the range of regular lookouts (an open port too near the water line, signs of shallows or impending collision—anything endangering the common interest) thereby partly forfeit their claims to safe passage. The writer is deeply concerned about the field of hydrologic and related scientific research, particularly regarding,

(*a*) Thorough utilization of existing basic data;
(*b*) Expansion of procurement programs to broaden their scope;
(*c*) Recognition of patterns among natural phenomena, capable of affording guidance in supplying reasonably close approximations of missing basic data; and, finally,
(*d*) Simplification of procedures whereby the divergent factors may be made to yield an integrated, logical, and conclusive result, capable of withstanding repeated practical tests consistently and satisfactorily.

The writer's first glance at Fig. 14 disclosed what appeared to be a break in the continuity of the series between the curves for 30 and 120 min. Obviously, a 60-min curve (intermediate yet not quite midway between the adjacent graphs) should have been included in Fig. 14. This would appear as a straight line connecting the origin of coordinates with the abscissa value of 4.5 at the top margin. By the very nature of this relation, numerical values of ordinates and abscissas at all intersections on the grid for the 60-min curve will be equal. The fact that the hourly rainfall intensities are displayed along the vertical axis was not overlooked.

An additional curve that might well be drawn in Fig. 14 concerns the 24-hr period. Assuming that the only rainfall during such a period occurs within

the 240-min (4-hr) interval which the curve represents, the average hourly intensity would be $\frac{4 \times 4.5}{24} = 0.75$ in. Therefore, the new curve or straight line would connect the origin of coordinates with the point on the top margin corresponding to abscissa 0.75, representing the extreme limit toward the vertical axis. Allowing for the gradual increase of precipitation prior to the beginning of the 4-hr period of high intensity and for a corresponding decrease later, as well as for the associated showers which combine to make up the 24-hr maximum for each designated frequency within the series, the final position of the graph would connect the origin with the value of approximately 1.0 along the top margin. This result is logical because 24-in. rainfall depths have been attained rarely, if ever, within the scope of officially accredited records in the United States. The most outstanding example to be found in the writer's investigation (first reported in 1930), covering a period of some five years but dealing with several times as many stations as were included in the final tabulation (39), was the record at Taylor, Tex., of 23.1 in., associated with a maximum hourly rate of 4.25 and a monthly maximum of 25.9 in. Similar maxima recorded for Austin, Tex., in another tabulation of more recent date are 19.0, 4.38, and 20.8 in., respectively (40).

If maximum observed values are desired for some of the world's outstandingly high rainfall stations, one must include 24-hr depths of 21.9 and 28.7 in. for remote stations in Hawaii and Jamaica, British West Indies, respectively, and 25.0 in. for Madras, India, and 40.0 in. for Cherra Punji, India, associated with hourly maxima of somewhat less than 4.0 in. and with daily and monthly maxima of 42.8 and 360.1 in., respectively (39)(40).

Similarly, for the Baguio Station, high among the mountains of Luzon, Philippine Islands, the maximum hourly rate was nearly 4.0 in., whereas the maxima for daily and monthly periods were recorded and published as 46.0 and 133 in., respectively. Obviously, the curve for century-frequency maximum intensities that includes all known values should connect a point on the top margin with abscissa nearly equal to 2.0 in. per hr with the origin; it would therefore lie slightly to the right of the curve of Fig. 14, representing the maximum 4-hr intensities within the United States. The last-described curve, dealing with world maxima, should preferably be excluded, since it would disturb the orderly march of the series developed for the United States.

As far as the writer was able to determine, much of the concept relating to rainfall patterns was new when it was published in 1930 (39). Two illustrations (39*a*)(39*b*), a table (39*c*), and textual references pertaining to all three incline toward the establishment of a nearly straight-line relationship for the series. For many of the stations listed (39*d*), comprising 820 of the principal station-records throughout the world in summarized form, the writer reported that the mean annual rainfall divided by the average number of rainy days per year would represent the maximum rainfall depth to be expected within a 3-min period (39*e*). Obviously, this curve would lie somewhat to the right of the one in Fig. 14 and might well serve as an envelope of the plotted values below the graph for a 5-min duration. Inasmuch as the majority of the stations listed by the writer (39*d*) for world-wide distribution had no data for

periods shorter than one day, the intermediate values between the 3-min period and the 24-hr period had to be determined by interpolation. For this purpose, the writer determined by trial that curves for the 15-min and 60-min durations practically trisected the space between the 3-min and 24-hr maxima. For that reason, principally, the 30-min rainfall intensities were purposely omitted, as being less significant members of the series than were those retained. Other members of the plotted series of rainfall data (39*a*) were maxima for both monthly and yearly rainfall depths. Differences in lengths of records for the respective periods, from a day or less to months and years, were inclined to disturb the true relationships. It was not uncommon to find only a decade or so of rainfall intensity records for a day or less, whereas the monthly and yearly values as published covered half a century or more, thus calling for some adjustment to reduce such data to a basis of comparable frequencies.

Under the heading, "General," Mr. Hathaway states:

> "A study of rainfall intensity-frequency data for a large number of precipitation stations indicates that there are fairly consistent relationships between the average intensity of rainfall for a period of 1 hr and the average rates of comparable frequency for shorter intervals, regardless of the geographical location of the stations or frequency of 1-hr rainfall."

This terse statement is of far-reaching importance and somewhat startling on first reading, as it links hourly and shorter intensities regardless of geographical location and frequency of each series. It proves to be almost axiomatic in the light of further study. In fact, it is a part of the concept of rainfall patterns which the writer sought to express in 1930 (39) and earlier. In an effort to define the essential similarities inherent in precipitation intensities or total depths for varying periods of time from 5 min to a year, the writer expressed all these quantities as percentages of the mean annual depth of rainfall. Thus, the maxima for a 5-min period ranged from 0.17 in. at Juneau, Alaska, to 0.91 in. at Denver, Colo., or to 1.05 in. at Tampa, Fla., or to 1.15 in. at Terre Haute, Ind., representing the maximum officially reported and observed depth in the United States. Reduced to percentages of annual mean rainfall depths for the respective stations, the maxima become 0.2%, 6.4%, 2.1%, and 3.0%. Some other stations have established claims for somewhat higher records but have not fully qualified them as official data. Undoubtedly still higher values have occurred for tropical storms, although official observations are lacking.

Isohyetal Maps.—For several years after the publication of technical reports, including the volume entitled "Storm Rainfall of Eastern United States" (41), by the Miami Conservancy District, many hydrologists considered that the Great Plains and Western States were not susceptible of the same treatment. The reason for this opinion was related to the erratic and unpredictable rainfall occurrence, amount, intensity, and areal extent, as well as to the dominant influence of altitude over latitude. In the face of such professional opinion, the writer undertook to extend a series of isohyetal lines representing not only daily but also monthly maxima throughout the United States (39*f*) (39*g*), and then completed the maximum monthly isohyetals for such parts of the various continents as were covered by officially published records. Mr. Yarnell (15) adopted scales that precluded the portrayal of local effects, such

as are associated with topographic forms, rain shadows, normal paths of air-mass movements, and the orographic barriers to be surmounted. To overcome this difficulty in part, Mr. Yarnell introduced special maps on a more generous scale, covering the Pacific Coastal States, with regard to 24-hr rainfall depths for various frequencies, from 5 to 100 years. Lack of data had been noted on the Pacific Coastal strip for some of the shorter periods—namely, 16, 8, and 4 hr—and by implication for still shorter periods.

The writer's experience in the construction and drainage of arsenal and airfield facilities lead him to offer the following observations:

(*A*) No matter how liberally one may interpret the series of isohyetals on the small maps of Fig. 13, there is urgent need for either supplemental maps of the Pacific Coastal region on scales permitting the portrayal of the pronounced local variations, or a special note of caution concerning the practical certainty that the indicated limit of 1.25 in. per hr has been and will be greatly exceeded within 25-yr periods of record.

TABLE 9.—SUMMARIZED RAINFALL RECORDS OF THE PACIFIC COASTAL REGION

Item No.	Station	Altitude (ft)	Maximum[a,b]			Annual Rainfall[b]		
			Hour	Day	Month	Minimum	Mean	Maximum
	Washington:							
1	Tatoosh Island..	86	1.44	5.91	25.8	49.6	83.5	114.3
2	Forks[c].....	480	(2.3)	(6.6)	31.3	108.4	120.3	153.8
3	Spruce[c].....	300	(2.0)	(6.0)	24.6	79.6	124.4	141.0
4	Quinault[c].....	300	(2.4)	(7.0)	42.3	79.2	129.5	157.1
5	Seattle.....	246	0.80	3.52	15.3	20.0	33.8	56.4
6	Tacoma.....	109	0.79	3.79	18.9	22.0	39.9	57.3
7	North Head....	211	0.86	4.29	17.8	39.9	55.5	78.6
	Oregon:							
8	Tillamook[c].....	23	(2.2)	(6.5)	37.2	63.6	95.9	132.1
9	Nehalem[c].....	20	(2.5)	(7.0)	29.3	82.5	109.3	134.4
10	Glenora[c].....	575	(2.7)	(8.0)	50.2	97.3	130.5	167.3
11	Portland.....	57	1.31	7.70	20.1	30.8	42.8	67.2
12	Roseburg.....	510	1.14	4.10	12.8	21.1	33.3	44.3
	California:							
13	Monumental....	2,700	(3.0)	(9.0)	43.8	69.3	109.1	153.5
14	Crescent City...	125	(2.6)	(8.0)	31.9	47.5	74.8	107.6
15	Eureka.....	64	1.30	6.87	19.5	22.4	41.8	64.5
16	Red Bluff.....	307	1.72	6.10	20.7	12.9	24.6	49.0
17	Point Reyes....	490	1.40	3.10	11.4	7.3	20.6	32.4
18	San Francisco...	206	1.07	4.70	24.4	9.0	22.5	38.8
19	Sacramento.....	71	0.94	7.20	16.0	7.8	18.3	34.9
20	Fresno.....	295	1.36	2.90	9.5	3.9	9.7	21.5
21	Tamarack.....	8,060	(2.8)	(8.5)	39.8	27.8	50.6	94.3
22	Los Angeles....	361	1.51	7.36	15.8	4.8	15.6	40.3
23	San Diego.....	87	1.16	3.20	9.3	4.3	9.8	27.6
24	Campo.....	2,543	(4.6)	(8.6)	20.4	11.3	19.8	35.3
25	Squirrel Inn....	5,280	(3.3)	16.7	30.3	19.6	40.8	82.2
26	Scales.....	....	(2.0)	(6.0)	29.0	49.8	87.6	125.5
27	Kings Canyon..	....	(3.0)	(5.0)	8.6	....	38.4	....

[a] Parentheses enclose unofficial or estimated quantities. [b] All units are in inches. [c] Northwest.

(*B*) A study of the selected station-records in Table 9 (which were chosen intentionally to represent the upper bracket of observed intensities, inasmuch as most of the actual maxima of high mountainous regions are unrecorded) indicates that the influence of latitude is subordinated to that of altitude combined with windward exposure.

(*C*) The seaward slopes of the coastal ranges extending from the Olympic Peninsula in Washington to San Diego, Calif. (and probably even farther) and from the western slopes of higher ranges farther inland (such as the Cascade Range and the Sierra Nevada) are capable of developing a fair approach to the series of intensities plotted around the Gulf Coast. These intensities approach about 4 in. per hr at outstanding stations, although they are ordinarily limited to roughly half of that depth for available periods of hourly record.

(*D*) In Table 9, the rainfall patterns disclosed by neighboring stations are used for practical purposes to estimate appropriate values of the missing items of basic data. All these are enclosed in parentheses to indicate their unofficial character. These estimated values may be subject to uncertainties of approximately 20%.

(*E*) Both limiting and periodic intensities for selected rainfall stations seem to vary with the square root of the number of minutes in the respective time intervals considered, as has been stated in official publications (42)(43). Thus, the envelope of conceivable intensities—perhaps far beyond the attainable limit for most of the outstanding stations of high rainfall—would have depths for respective periods as follows:

Minutes	Inches
1	1.00
5	2.24
10	3.16
15	3.88
30	5.48
60	7.75
240 (4 hr)	15.50
1,440 (24 hr)	37.90
43,200 (month of 30 days)	207.4
525,600 (year of 365 days)	725

Some of the foregoing depths have been attained or even exceeded—for example, the daily, monthly, and yearly depths at Cherra Punji and the daily and monthly depths at Baguio. Some have been either attained or closely approached at several stations in India and the tropical islands of the Pacific Ocean. During seven of the fourteen years of available record for Cherra Punji (1905 to 1919), the maximum daily rainfall depths were in excess of 20 in.; and three of these exceeded 30 in., as follows: 33.00, 31.05, and 32.35 in. during 1910, 1914, and 1915, respectively. Desert regions may attain a much greater proportion of the short-period rainfall intensities, say, for less than an hour, than should be expected for days or months. Otherwise, they would cease to be classified as deserts, inclining more toward the humid in character. Patterns for most stations will follow more or less closely some proportion of the foregoing envelope depths.

Runoff Formulas.—In the interest of simplified procedure and computations, Mr. Hathaway has wisely made a number of assumptions, some of which are admittedly only approximations to cover variable conditions and behavior. (Either compensating or cumulative errors may result for a given problem.)

Among these are the assumed constancy of rainfall, infiltration, and resulting rate of supply for overland flow during a specific storm. Inasmuch as the assumed rectangular form of pluviograph is seldom if ever attained and an alternate form of isosceles triangle on the same time base but with its altitude double the height of the rectangle would enclose an equal area and would measurably alter the proportions for infiltration and supply volumes, this alternate form of pluviograph may bear investigation. Although it would still be an approximation, it would occupy a mean instead of an extreme position.

Mr. Horton's equations for overland flow, as presented by Mr. Hathaway, as well as a single page of textual material in Mr. Yarnell's notable bulletin (15), employ the so-called "rational equation" for runoff, $Q = C\,i\,A$, which expresses runoff Q in cubic feet per second, representing some percentage or decimal fraction (C), the ratio of runoff to rainfall for the critical period, times the product of rainfall intensity i in inches per hour and the number of acres A in the drainage area. The infiltration capacity is likewise expressed in inches per hour; as, for example, 0.6 in. per hr adopted by Mr. Hathaway for turfed areas. Similarly, the allowable pond storage of 2,000 cu ft per acre in the example used by Mr. Hathaway may be expressed as a depth of 0.55 in., since an acre-inch is $\frac{43{,}560}{12} = 3{,}630$ cu ft.

For the example solved by Mr. Hathaway according to the rigorous procedure recently developed with discharge to the inlets of the drainage system given as 2.1 in., from a rainfall intensity of 3.6 in. per hr, C is equal to $\frac{2.1}{3.6} = 0.583$, or 58.3%. Evidently, the surface storage outside of ponds was 0.35 in., which added to $0.60 + 0.55 + 2.10 = 3.60$. Only that portion of the surface storage (10% of the total rainfall) is subject to estimation; it could not be seriously in error; if neglected entirely, the error would be only 10%.

Perhaps it is just plain coincidence, but it is interesting to note that the X-ratio value derived graphically by Mr. Hathaway is 59% (compare with 58.3%), representing the ratio of drain-inlet capacity to peak rate of runoff, the difference being absorbed in pond storage.

The impressive array of well-chosen tables of basic data and related computations, together with the charts and diagrams on which the illustrative solutions of typical problems are readily traceable, and the clear-cut, logical treatment of the entire procedure in the accompanying text by Mr. Hathaway leave but little to be supplied in discussion. It is gratifying to know that the design procedures have already been incorporated in the Engineering Manual of the Construction Division, Office of the Chief of Engineers (23).

The susceptibility of the entire analysis and rigorously derived results to comparison and expression in terms of the so-called "rational formula," as demonstrated herein, should insure their incorporation in professional practice much more widely than would have been possible without such a popularly established device. This formula results in closing a survey circuit well within the tolerance limits, when results derived, or capable of derivation by both methods, are found to be in substantial agreement. Mr. Hathaway's paper represents a noteworthy accomplishment and progress in the field of hydrometeorology.

D. D. McGUIRE,[16] ESQ.[16a]—A splendid method for rational design of drainage capacity is contained in Mr. Hathaway's paper. The engineer is given a measuring stick by which he may correlate the effect of the storm deemed critical for a particular installation with the airport topography and the drain capacities necessary for the required use of the airfield. Such a collection of concise data and workable information is appreciated by the drainage engineer.

Proper use of subgrade drainage may often simplify surface drainage facilities. Where extensive flat areas are involved, a better over-all solution of the problem may sometimes be obtained by a combination of several methods. The following analysis and comments give some reasons for serious consideration of subgrade drainage facilities.

The 1-hr infiltration value subtracted from the 1-hr rainfall quantity identifies the standard supply curve and forms the basis for succeeding computations. In the example quoted in Mr. Hathaway's computation of storm drain capacities assuming supplemental ponding, the design storm chosen presumes 2.2 in. of rainfall per hour. The 1-hr infiltration capacity for turfed areas is assumed to be 0.6 in. per hr and represents 36.7% of the rainfall quantity. Surely this is important enough to warrant further investigation and more accurate data on infiltration capacities. The pertinent factors are many and their interrelationship apparently vague. Problems presenting greater difficulties have been solved previously through research.

If 36.7% of the rainfall quantity may be disposed of through infiltration, why not either dispose of the water quickly through subdrainage facilities and thereby reduce the amount of ponding, or, by lowering the adjacent water table, create a ponding area below the surface and away from traffic.

The possibilities of subdrainage pipe seem wide and valuable. Quick disposal of filtered water will reduce overland flow and minimize surface erosion. The subdrainage system creates a storage area below the plane of traffic, thereby reducing the number of required inlets with their attendant safety hazards.

Lowering the water table reduces the damaging action of frost on subgrades under runways and aprons and, through better stabilization, can reduce the cost of surface treatment. A proper subdrainage system will make available for landing outside of runways areas which might never attain a usable condition if the entire disposal depended on inlets and storm sewers.

In view of the many facets of the subdrainage problem, it is hoped that the same clear, usable type of analysis described by Mr. Hathaway for storm sewer disposal systems may be evolved for the design and efficient use of subdrainage pipe.

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[16a] Received by the Secretary May 8, 1944.

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[17a] Received by the Secretary May 8, 1944.

should find this important and timely Symposium indispensable. It contains many sound principles of utmost value to the designer of airfields, presented in a simple, straightforward, and stimulating manner.

In connection with airfield design and construction problems, the Corps of Engineers has proceeded quietly and systematically in designing, constructing, and testing, to meet adequately the demands of the airfield war construction program. Although greater and greater wheel loads have been predicted, comparatively little information has been published on methods proposed to meet such an airfield problem.

Suitability of Base-Course Materials.—In discussing base courses for concrete pavements, Colonel Stratton uses the term "high-quality base materials" without sufficient definition. Because of a common misunderstanding as to what constitutes a "high-quality base material" and because of the belief of many engineers that clay should be added to granular materials for use as a base course (simply because the satisfactory performance has been observed of well-graded sand or gravel mixtures containing clay when used as surface courses exposed directly to light wheel loads without the use of substantial pavements), it should be stated that, in general, clean, well-graded granular base materials with a minimum of "fines," say, not over 4% or 5% passing No. 200 mesh, make the most desirable high-quality bases for both rigid and flexible pavements.

Only 3% by weight finer than 0.02 mm in a well-graded granular material used as a base course in northern areas may cause troublesome frost action—breaking of concrete due to irregular heaving in winter and insufficient support due to softening in the spring.

In southern regions, natural clayey gravel or sand-clay mixtures may give satisfactory results if placed in a properly compacted, homogeneous condition. However, the deliberate addition of clays to clean, well-graded granular materials for use as base courses should be discouraged, not only because of the difficulty, expense, and length of mixing process, but also because of the inherent increased sensitivity to moisture control for proper compaction, as well as because of the increased potential susceptibility to subsequent moisture changes.

Design of Concrete Pavements.—War conditions have restricted the use of steel which in turn has caused the revision of concrete pavement sections. More than one good engineer pleaded that the reinforcing steel be left in pavements, and especially those which had to be placed directly on poor subgrades without the benefit of intermediate base courses.

There is no question about the necessity for thickening the edges of concrete pavement as is done in the revised sections if the pavement is placed directly on a poor subgrade. However, it would be very desirable to make additional investigations as to the necessity for thickening any edges when the pavement is placed on a high-quality, well-compacted base course, considering the minimization of differential settlement, together with the loading frequency and the natural traffic distribution on airfields. Is it not probable that the effect of disturbing the base course in shaping for thickened edges counteracts many benefits obtained from the thickened edges?

Control of Frost Action.—In discussing pavement design for frost action, Colonel Stratton states that the most generally accepted method of insuring pavement or base-course stability over a potential frost-action soil is the provision of sufficient thickness of insulating material.

The writer would question the mere use of "insulating material" over a potential frost-heaving soil to insure stability unless the term "insulating material" were defined as a granular material containing no more than a small percentage of material finer than 0.02 mm. Without such a restriction, one can well imagine a granular material with, say, 10% finer than 0.02 mm which, if it were possible to keep dry, would act as an insulating material but which, because of its high percentage of "fines" and high potential capillarity, might cause serious damage from frost heave if used to cover another potential frost-action soil in contact with a water supply. Therefore, in using soils as insulating materials, it is necessary to place a limit, not necessarily on the grain-size curve itself, but on the allowable amount of soil finer than a certain size, unless the soil can be entirely separated from a water supply either by using an intermediate impervious course or by so selecting the soil that the limit of capillary rise will not fall within the zone of frost penetration.

Subsurface Drainage.—In the past much time and money have been wasted in installing subsurface drainage systems which, as Colonel Stratton states, could frequently have been eliminated by providing an adequate and properly designed surface drainage system.

A subsurface drainage system is referred to herein as a system installed primarily for the purpose of removing water from pavement base courses or subgrades—some form of improvised "French drain," a system of open-jointed pipe, or an assemblage of well points.

When studying subsurface drainage, it is expedient to classify soils in one of the three following groups: (*a*) Well-draining soils such as clean sands and gravels; (*b*) Poorly-draining soils like the slightly plastic organic and inorganic silts; and (*c*) Impervious soils—for example, homogeneous clays. (These remarks on drainage are based on lectures by Arthur Casagrande, M. Am. Soc. C. E., on the "Control of Soils in Airport Construction," delivered at the Graduate School of Engineering, Harvard University, Cambridge, Mass.)

Colonel Stratton states that "* * * where the ground water is above the frost line or causes soft, boggy conditions in areas where soil stability is required, subsurface drainage is necessary." It should be noted, however, that (*a*) lowering the water below the frost line may or may not prevent frost heave, depending on the height of capillary rise, and (*b*) it may be much easier and more practicable in some cases to lower the water table by using circumferential ditches or by providing drainage through an underlying impervious stratum when such a favorable condition exists.

For clean well-draining granular materials, subsurface drainage can often be eliminated altogether, or at least reduced to a minimum by providing circumferential ditches to intercept ground water from surrounding areas and by sealing off the surfaces to reduce the infiltration of water into underlying courses or subgrades. In other words, the water should be kept from entering the base courses, but, if some does penetrate, the perviousness of these courses should be

advantageous in removing the water. If the well-draining materials contain an appreciable amount of fines, then, to prevent possible serious damage from frost heave, it will be necessary to keep the water table not only below the frost line but at a distance below the frost line greater than the height of capillary rise for the particular soil.

In a strict sense, the statement that poorly-draining soils are "drained" with a subsurface drainage system is incorrect. The height of capillary rise in such soils is usually great, and as long as the water table lies within the height of capillary rise the soil will remain saturated by capillary water. If the frost zone comes within this zone of capillary water, there may be danger of serious frost heave. Therefore, there is no advantage in attempting to use subsurface drains to eliminate danger of frost heave in poorly-draining soils, unless the drains can be placed at such depths that the piezometric water table will be lowered to a distance below the frost line greater than the height of capillary saturation. For some soils, this is impracticable or impossible. However, benefits can be derived from using subsurface drains in poorly-draining soils. By their use soft soils can be consolidated and thereby their bearing value can be increased.

In using subsurface drains for consolidation purposes the following should be emphasized:

(*a*) Since any benefits of consolidation are obtained from capillary forces (that is, as long as the soil remains completely saturated), the deeper the drains can be placed within the zone of capillary saturation, the greater will be the force producing consolidation.

(*b*) The closer the drains can be placed for a given depth, the less time is required to consolidate the soil between the drains.

(*c*) To function properly the system must be so designed that water can be removed as fast as it collects. It may be possible to accomplish this with a gravity system in certain favorable locations, but, more often, it will be necessary to use pumps in sumps. Pumps are also needed when well points are used.

(*d*) Drain pipes for such a system may be very small since the actual amount of water moving through them may be no more than a "trickle." (The overlying soil remains saturated by capillary water so that the greater part of water from that source will be only that which is squeezed out by consolidation.) Since the soil by nature drains poorly, only a small quantity of water can be expected from infiltration and ground-water movement.

(*e*) The ideal situation would be one in which the poorly-draining soil is underlain by a pervious layer, in which case water could be removed from the pervious layer by means of well points or circumferential ditches. The action of a subsurface drainage system in homogeneous clay is so extremely slow that it is of little value for improving the condition of the clay.

Construction Methods.—Good designs may be vitiated by poor construction, and it should be stressed that high-quality construction control and inspection are essential to many types of construction.

The value of "engineered" subgrades and bases is more and more apparent. An excellent pavement may deteriorate on an inferior base and subgrade,

whereas a second-grade pavement on a properly designed and constructed foundation may give excellent service. An engineered foundation requires proper control from design through completion. It requires careful selection and placing of materials, proper moisture control, and adequate compaction—the real value of which cannot be overstressed. Many of the defects of settlement, swelling, and shrinking, so often blamed on the pavements, can be traced to the foundations and could have been minimized by proper compaction control.

Remarks on Training of Soils Engineers.—In teaching Army courses on "The Control of Soils in Military Construction," the writer stresses particularly the value of identifying soils correctly because thereby the possibility of making a major mistake may be eliminated. After a soil is correctly identified, the properly trained soils engineer and his inspectors should know how to improve that type of soil, how to judge its drainage characteristics, how to decide the degree of moisture control necessary for efficient compaction, and how to determine what method should be used for compaction.

In prewar days, too little attention was given to the proper training of field men and inspectors for all types of highway and airfield construction. Frequently proper control and inspection was sacrificed for impressive looking plans. If the postwar period is to bring extensive highway and airfield construction with ample available man power, as is anticipated, then surely something will have to be done to assure not only the proper utilization of experienced personnel returning to civilian work, but also the provision of ways and means for the training of additional men. The time to consider these problems has already arrived.

A. CASAGRANDE,[18] M. Am. Soc. C. E.[18a]—The construction of transportation facilities—roads, bridges, railways, and waterways—has been an age-old source of problems to the civil engineer, and now the construction of airports, with a scope undreamed of only a few years ago, may well be making an important new chapter in civil engineering. When such rapid developments take place, reliable published information often lags far behind. This Symposium is a fortunate exception.

The following discussion deals primarily with the first paper of the Symposium. Colonel Stratton has succeeded remarkably well in summarizing all pertinent information and essential considerations on this subject, with particular emphasis on the many important contributions which the Army Engineers have made to the design and construction of airfields for heavy plane loads. Although restricted in scope to military airfields, it should be emphasized that most of the Symposium also applies to the commercial airfields of the future. In fact, almost all of the extensive research on the design and construction of airfields, conducted under the direction of Colonel Stratton, will be of inestimable value for the postwar development of commercial air facilities to serve the air liners of the immediate future. Therefore, it is hoped that this paper will be followed by others describing the results of the large-scale traffic tests which

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[18a] Received by the Secretary May 15, 1944.

have been undertaken by the U. S. Engineer Department, the results of which will be indispensable to every engineer designing pavements for heavy planes.

Design of Flexible Pavements.—The design procedure adopted by the Office of the Chief of Engineers is based on the empirical method developed by the California State Highway Department. As stated by Colonel Stratton, the design curves of Fig. 3 have been confirmed by large-scale traffic tests.

It may be well to explain why it has not yet been possible to develop a successful theoretical approach to the design of flexible pavements, based on the theory of elasticity. Perhaps the most important reason is that the coarse-grained materials, of which pavement and base courses are generally composed, do not follow Hooke's law. Therefore, the distribution of stresses in the subgrade will show considerably greater concentrations than that computed by the theory of elasticity. In addition, most flexible pavements do not have enough tensile strength to resist the considerable tensile stresses which develop, according to the theory of elasticity, for large ratios in the modulus of elasticity of pavement-plus-base over that of the subgrade. Even if no tension cracks develop, the computed stresses in the subgrade would be on the unsafe side because the deformation of the pavement in tension is much greater than that in compression; in other words, for practical purposes the pavement cannot take tension. These remarks are not intended to discourage further attempts at theoretical studies of the design of flexible pavements, but merely to emphasize difficulties which are often overlooked and which can lead to conclusions very much on the unsafe side.

Stabilized Versus Compacted Base.—Much confusion still prevails on the value of admixture of "fines" or "binder" to a clean coarse-grained material in the construction of base courses. To cite two extremes: (1) Excellent concrete pavements have been built in Long Island on hydraulically placed, clean, beach sand, without even an attempt at compacting the sand; whereas (2) in a different place, well-graded, fairly clean, glacial gravel (the best base material next to crushed stone) was "stabilized" by adding 15% material passing No. 200 mesh, thereby changing it into a frost-heaving material. Even where there is no frost action, the addition of a plastic binder to gravel has often resulted in a type of base which, in a condition of capillary saturation (evaporation from its surface prevented by the overlying pavement), was much less resistant to deformation than the coarse material without the binder. Not only is it costly to mix a binder with coarse material, but it is difficult to make the mixture uniform. Even a small excess of clay binder may cause local destruction of a pavement, which would not happen if the admixture of such binder were omitted.

Instead of paying for the admixture of fines, time and money should be directed toward very thorough compaction of the base-course materials. Although it has been emphasized, far too little attention is paid to proper compaction. In fact, some engineers really believe that a uniform sand cannot be compacted; that it always remains loose. (Perhaps this belief is due to the ambiguity of the term "loose," which is also used to designate grains with no cohesion.) The writer foresees the development of compaction equipment which by means of vibration will produce degrees of compaction of coarse-

grained soils unattainable with any methods now used, and at little or no additional cost.

The importance of thorough compaction is best illustrated by the fact that a well-compacted sand has about three times more resistance to deformation and bearing capacity than the same sand in its ordinary loose condition (not bulked). For a well-graded gravel this ratio may be as high as fifteen. The large increase in resistance to deformation of gravel surfaces exposed to constant traffic is well known. Such surfaces may not be satisfactory unless traffic compacted. Does this not prove that it is possible to compact such materials far better than by ordinary construction methods? What can be done by traffic can be accomplished by vibration equipment especially designed for the compaction of sandy and gravelly soils.

Considering that the cost of compaction is only a fraction of the cost of the material itself, particularly where long hauls are necessary, it is only logical to improve the material by compaction to the utmost limit. Manufacturers of construction equipment would do well to pay attention to this problem and to its partial solution already attained before the beginning of World War II (44).

Design of Concrete Pavements.—Colonel Stratton advances pertinent arguments for restricting the thickness of concrete pavements to about 8 in., and for providing the necessary support on poor subgrades by an appropriate increase in the thickness of the gravel base. Success of such a design will depend on the uniformity of subgrade support. When dealing with nonuniform support such as results from irregular subgrade conditions, soils subject to swelling and shrinkage, or nonuniform frost action, in spite of a good base at times there may be little support, and corners may break under the design load, or even less load. For such unfavorable subgrade conditions, the use of thicker slabs with steel reinforcement cannot be avoided. Warping stresses might be controlled by a sufficiently small size of the slabs, say, 12 ft by 12 ft for a pavement 12 in. thick. The wire mesh reinforcement will preferably be used in two layers, spaced about 3 in. from the top and bottom of the slab. Some additional reinforcement in the top of the pavement should be provided at the corners. In such a design the steel would actually participate in carrying stresses, instead of acting merely to tie broken slabs together.

In thick pavements even greater attention will have to be paid to the design of the joints for very heavy loads. The most desirable solution would be to make all joints alike, each one permitting a slight amount of expansion. A trapezoidal tongue and groove design could be used with premolded asphaltic boards, about 3/16 in. thick, containing enough air voids (for example, in form of admixture of cork particles) to permit a compression of about 1/16 in. Thus, the total expansion possible for 100 ft of pavement, assuming a 12-ft spacing of the joints, would be about 1/2 in. This allowance is ample in view of the uniform distribution of this expansion and the greater assurance that all joints will remain operative, as compared with the standard, wide expansion joints that have been known to become partly inoperative with time. Heavy, short steel dowels should be provided, painted with asphalt and having a thin compressible disk glued to each end, to allow for the necessary slight amount of movement. If vertical steel pins are used to hold the premolded boards in place (piercing the

trapezoidal tongue and being driven into base or subgrade), the section inside the tongue may require a sleeve or cover to allow movement between this part of the pin and the concrete.

A uniform joint construction, distributing expansion uniformly over all joints, would aid greatly in solving many undesirable features in the concrete pavements of the past.

Insulation for Frost Protection.—One of the feasible methods for protection against frost action is the use of an insulating material beneath the pavement, of such thickness as to prevent frost penetration into the subgrade. Where cheap timber is available and gravel scarce, timber deserves serious consideration, because a thickness of 3 in. will offer the same protection as 18 to 30 in. of gravel.

Another material which the writer has seriously considered for this purpose, ever since he became acquainted with it in 1929, is a porous concrete made from a mixture of cement mortar and foam. This Danish development has been used in Europe and South America for heat insulation and other construction. Fracture surfaces have the appearance of slag. It can be manufactured in different unit weights and insulating characteristics, and has the important advantage that it can be mixed in the field and poured like concrete. This, or some other cheaply made material full of air voids and having a compressive strength of at least 300 lb per sq in. (which will not soak up water into the air voids when submerged), poured either in place or in pre-cast slabs, would solve the problem of frost action in cases where the necessary thickness of gravel would be prohibitive in cost. In areas of permanently frozen ground such insulating materials could serve to protect the frozen ground under pavements from melting.

Wheel Loads of Airplanes.—Airplane designers should realize that placing the entire weight of the heaviest planes on only two wheels, or on two pairs of dual wheels with little space between the dual tires, creates the necessity of very expensive airfield pavements and limits the number of airfields on which such planes can land safely. For the heaviest planes the load should be spread over four wheels, with the dual tires spaced at least 6 ft apart. For further increase in plane load to 150 tons, the writer foresees the necessity of spreading the load over two four-wheel carriages, resembling perhaps the wheel carriages of railroad cars. With such an eight-wheel landing gear, a 150-ton plane would have the enormous advantage that in an emergency it could land on an ordinary commercial airfield without danger of breaking through the pavement and of serious consequences for plane and passengers. Civil engineers specializing in airfield construction should be consulted in connection with the design of the landing gear of air liners.

Closing Remarks.—The writer takes this opportunity to express his appreciation of the broad application of soil mechanics to the design of earthworks, foundations, water-front construction, and now to airfields, which the Army Engineers have followed during the past decade, and in which Colonel Stratton has played a leading rôle. The Army Engineers have contributed more to the advancement of applied soil mechanics than any other governmental or private organization.

HARRY E. COTTON,[19] M. AM. SOC. C. E.[19a]—The Corps of Engineers, with which the two authors (Colonel Stratton and Mr. Hathaway) of the Symposium under discussion are connected, has contributed much to the various problems of airport design and construction—especially base and subgrade design, proper backfill for subdrains, and runoff criteria for the design of storm drains.

Backfill for Subdrains.—The design of base-coarse drainage in Fig. 8(*b*) of Colonel Stratton's paper shows the perforations placed up which is contrary to good practice especially if proper backfill (filter material) is used. This suggests that a discussion is in order on backfill, the proper design of the infiltration area (perforations), and the amount of infiltration area required.

The chief enemy of subdrains is clogging either of the pervious trench backfill or of the pipe itself or of both. This problem has been solved in an investigation undertaken by the U. S. Waterways Experiment Station at Vicksburg, Miss. (45). The function of a subdrain is to remove ground water from a water bearing zone. Such zones are of previous material—silts, fine sands, coarse sands, and gravel. To prevent the entrance of the fine soil particles into the trench backfill, the investigation shows that the backfill should be not coarser than that in Table 10.

TABLE 10.—MAXIMUM SIZE FOR FILTER MATERIAL

Standard A.S.T.M. sieve	Percentage passing
3/8 in.	100
No. 3	86
No. 4	68
No. 6	52
No. 8	41
No. 10	36
No. 16	26
No. 20	18
No. 30	0

A clean backfill having a grain-size curve between the curve of grading in Table 10 and that of a medium sand (No. 60 sieve) will be satisfactory. This is a radical departure from the usual state highway specification for material mostly between 2 in. and 1/2 in. Some states allow 0 to 10% passing a No. 4 sieve and some allow material as large as 6 in. The idea that a coarse backfill is needed with a pipe subdrain probably originated with the requirements for a "French" drain (without pipe) in which large voids are necessary for lateral flow. The vertical flow capacity of medium sand is ample for all ordinary subdrain conditions. The pipe provides lateral flow.

Such fine grained backfill is not only necessary to prevent clogging but, in many localities, more economical. However, it is necessary that the drain pipe be designed to eliminate the entrance of the backfill and yet allow for the entrance of water.

Because most of the recommended backfill material is of a smaller grain size than the perforation diameter, it drops through the holes by gravity action when the perforations are in the upper half of the pipe. This fact was demonstrated in the Vicksburg tests as well as in all other similar tests previously conducted. Therefore, instead of the perforations being up, as shown in Fig. 8(*b*), they should be down, or in the upper part of the lower half of the pipe. Also, to prevent the backfill from entering, due to side pressure, the upper

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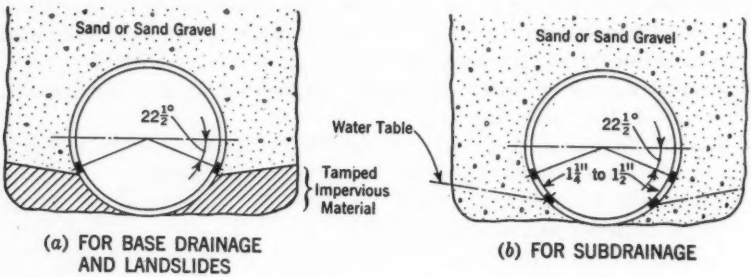
[19a] Received by the Secretary May 4, 1944.

row of holes should not be higher than 22.5° below the horizontal axis of the pipe as shown in Table 11.

There seems to be an undue concern among engineers regarding the loss of water once it has been collected inside the pipe. There are few instances in

TABLE 11.—RECOMMENDED PERFORATION DESIGN



Number of:	Section (a)	Section (b)
Rows of perforations (diameter, 3/8 ± in.)	2	4
Perforations per linear foot, averaged per section of pipe (minimum)	8	16

subdrainage work where such concern is justified. For the usual condition of lowering a water table in a saturated soil (section (b), Table 11), the zone under the pipe is already saturated and therefore no loss of water will occur. However, for landslide and terrace bank drains, to collect seepage that may be flowing on top of a thin impervious zone, the bottom of the pipe may lie in a

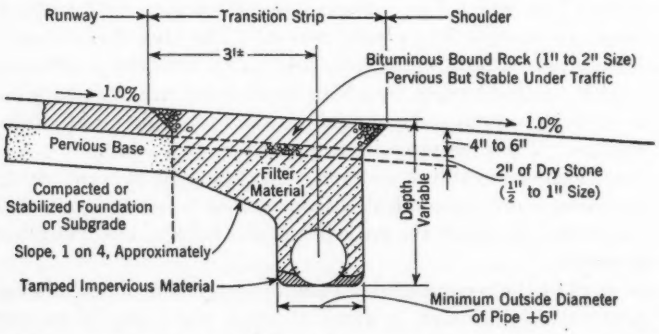


FIG. 38.—THE INTERCEPTING DRAIN APPLIED AS A COMBINED SURFACE DRAIN AND A BASE DRAIN

pervious soil and precaution should then be taken to prevent loss of water by methods such as those shown in section (a), Table 11. For these cases and for base drainage, the quantity of water to be collected is small and two rows of perforations are more than adequate. When drain pipes are installed adjacent to pavements in impervious soil zones for base drainage and also for intercepting

drains, an impervious trench bottom, as shown in section (a), Table 11, and Fig. 38, is advisable to insure complete removal of free water collected in the trench. The backfill or "filter material" over the top and sides should be a tamped mixture of sand and gravel not coarser than the grain-size curve of material in Table 10 but having a permeability sufficient for runoff.

Required Infiltration Area (Perforation Area).—The hydraulics of subdrainage is not an exact science. The engineer may be guided by research on ground-water flow, but the answer is largely based on judgment and experience. (Typical of such research are the runoff tests by the Engineering Experiment Stations of Iowa State College at Ames and the University of Minnesota at Minneapolis. Also the Darcy formula may be used:

$$Q = A\,k\,i \qquad (11)$$

in which k = coefficient of permeability, i = hydraulic gradient, and A = cross-section area of flow.) The ground-water flow is seldom sufficient to fill a 6-in. pipe within a 1,000-ft run. Even for the most severe seepage condition encountered in highway and railway cuts, 6-in. pipe has sufficient capacity in the first 400 to 500 ft of the run. A change of pipe size in the run (for example, 6 in. to 8 in.) is required to carry the accumulated interception but not because of increased ground-water flow to the drain per linear foot. Therefore, the larger drain pipes need no more infiltration area (perforations) than is required in a 6-in. pipe.

Little is known about the infiltration capacity of perforations in a pipe surrounded with sand or sand and gravel. One such test with a metal pipe gave an approximate answer of 0.009 cu ft per sec per sq in. of perforations (1/4-in. holes) for a condition in which little or no head was assumed above the top of the pipe. On this basis, a 6-in. pipe, on a 1.0% slope with perforations as specified in section (b), Table 11, could fill itself in less than 30 ft. When consideration is given to the fact that the area of perforations in a run of 16 ft as specified in section (b), Table 11, is equal to the end area of a 6-in. pipe, it is self-evident that the infiltration area is many times larger than is necessary; in other words, a large factor of safety is provided. When too many rows of perforations are specified in pipe, requiring the top row to be at or near the spring line (horizontal axis), solids are likely to enter. To prevent such entrance is one of the requirements of a well-designed subdrain pipe.

Impact.—Colonel Stratton's discussion of loading criteria leads the reader to conclude that impact is not a factor in the design of runway pavements and may even be construed to infer that impact does not occur. The subject of impact has probably received more discussion than any other one item in connection with airport design. The justification for designing runways for wheel loads without impact seems to be based mostly on the observation that runway pavements show less damage than apron pavements used for warming airplane engines. The writer is of the opinion that the apron failure is not due to increased load, but mostly to the failure of the foundation, which becomes puddled by vibration, creating a condition similar to that of beach sand when tapped with the foot.

It can be assumed that there is no impact at the instant of a perfect landing but, to this date, published data indicate definitely the existence of impact for the average landing. R. D. Evans, of the Goodyear Tire and Rubber Company, has stated (46):

> "In connection with this work [landing tests to determine the efficiency of tires in landings on a variety of soils and surfaces], we used two planes, one in the 3,000-lb. class and one in the 5,500-lb. class. They were equipped with a recording accelerometer whereby the actual impact factors developed during landing could be determined. As a result of this work, we arrived at the following conclusions: Ordinarily our typical landings made on hard surfaces developed impact factors ranging from 1.25 to 1.75 *g*. Anything less than 1.25 *g*. would be rated as an easy or soft landing and one above 1.75 or 2.0 *g*. would be called a hard landing. The hardest landing which we ever experienced in all this series of tests was 3.4 *g*. The design factor in both of these planes was, as I recall, 6.5 so that we still had quite a ways to go before approaching the limit of impact values which the planes were designed to withstand. In this connection, it may be of interest to point out that all types of airplane tires, if operated at recommended values of load and inflation will not 'bottom'; that is, deflect down to contact with the rim until the impact load reaches 3.5 to 4.0 times the static load xx."

In contrast to the Goodyear impact tests with light planes, the Douglas Aircraft Company conducted tests (47) on a DC-4 weighing 63,000 lb. The normal landing impact factor was 1.43 and the extreme landing impact factor was 3.5. Although there was a wide difference in the weight of planes used, the impact factors developed in both tests are in close agreement.

As a basis for the design of the primary airplane structure, the Civil Air Regulations (48) require, in addition to a factor of safety, depending on the part considered, a limit load factor for level landings and three-point landings. For planes as heavy as 2,000 lb gross weight this factor $n = 4.33$, whereas for planes heavier than 2,000 lb,

$$n = 2.8 + \frac{9{,}000}{W + 4{,}000} \qquad (12)$$

in which W = the gross weight in pounds. For a 74,000-lb plane (37,000-lb static wheel load) the factor n becomes 2.91, to which is added a factor of safety.

In summing up this impact question, it should be remembered that the ground reaction is the sum total of the forces causing tire deflection, shock absorber travel, and stresses in the chassis. Also, whenever the tire deflection or the shock absorber travel is greater than that under static load, impact must exist.

The California Bearing Ratio method is now used extensively for the design of airport pavements and pavement bases. This method is empirical, based on California highway experience, and no doubt contains an impact factor for highway traffic, the existence of which was so definitely proved by tests conducted at the Iowa Engineering Experiment Station. In these tests trucks moving as fast as 20 miles per hr were used, and transmitted loads were scale weighed. The resulting recommendation for the design of highway structures, including culverts, was that impact factors from 1.5 to 2 should be used. There-

fore, it would seem preferable to state that impact is allowed for in the California method rather than that no impact is used, or to infer that normal landings create no impact.

The time factor is another item that has received considerable discussion. The contention by some engineers seems to be that a load applied and removed suddenly does not produce the same transmission effect on the pavement foundation or on a buried structure as would a static load or a slow moving load. Whether this is true, or to what extent the transmitted load may be lessened by a supersudden load, as compared to just a sudden load, is not known. Observation has shown that mud pumping occurs from slab deflection under fast moving trucks as well as under slow moving trucks. Also the deflection of a large flexible culvert under a heavy railroad loading with 3 ft of earth cover has been observed. In this case a distinct deflection occurs with the passing of each of the front and rear group of wheels and with a deflection recovery between the application of these loads, thus indicating the speed with which moving loads are transmitted through earth.

Surface Drainage.—Mr. Hathaway's criteria for surface drainage is based entirely on catch basins or inlets. Two methods have been adopted for the collection of surface water—the catch basin (or inlet) and the surface intercepting drain. On some airports a combination has been used. Experience has shown that both methods have their advantages and disadvantages.

The "surface intercepting drain" was devised because of the apparent difficulty of concentrating surface water at one point without the use of pronounced gutters or ditches, which are objectionable on airport surfaces. The idea is to intercept the surface water without changing the uniform sheet flow to a piled-up flow; and, when properly functioning, these drains do just that. The two chief difficulties have been silting of the top layer and displacement of the top layer by traffic. These difficulties are not the fault of the idea, but are due to improper use and construction. Silting will occur when the surface water carries silts and fine sands; and, therefore, when used as indicated in Fig. 38, all turf areas should slope away from the drain. Regarding displacement, the tendency is to use too little bitumen. The correct amount can be determined best by making tests with the particular stone to be used. Contrary to the opinion of asphalt technicians, the claim is sometimes made that if enough bitumen is used to stabilize the surface, porosity is lost. This problem needs and deserves much more research work.

Catch basins (or inlets) function more satisfactorily on sumped areas than any other type. Those of ordinary size are not efficient in collecting water from valley surfaces where warping is limited as on airports. This fact is recognized by those who use extra long grate-top inlets across valleys in runway pavements. These valleys are generally located in the outer part of the pavement as shown in Fig. 8(*c*) to collect the runoff from the runway and thus prevent erosion and softening of the shoulder. Catch basins are even less efficient in collecting surface flow in turf valleys. A combination of the catch basin and surface intercepting drain system as shown in Fig. 39 would insure much better interception. For this use, the intercepting drain backfill

up to the surface should not be coarser than the grain-size curve of the filter material in Table 10. On newly graded fields, silting may, and probably will, occur, but, by using the recommended backfill, the silt and fine sands, carried by the surface water, will be stopped in approximately the upper 1 or 2 in. This can be bladed off and replaced with clean filter material. Under ordinary sod-growing conditions, such replacement will not be required more than once or twice.

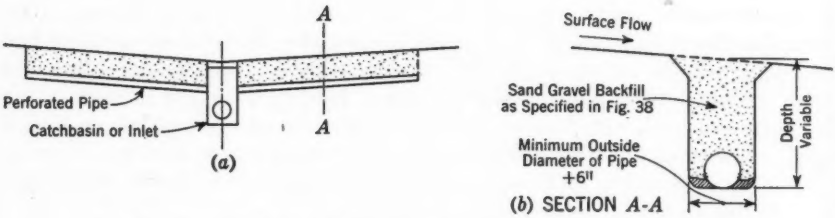


FIG. 39.—COMBINED CATCH BASIN AND INTERCEPTING DRAIN IN TURF VALLEYS

Combined Surface and Subdrain (or Base Drain) Systems.—The combined system, shown in Fig. 38, has received criticism on the ground that moisture may be introduced into the subgrade, or water may be forced into the pervious base, if and when the intercepting drain becomes full of water. For storms greater than that for which the drain pipe is designed (frequencies of 1 year, 2 years, 5 years, etc.), the trench backfill may become full of water, but theoretically only once in the design frequency period and then only for a few minutes. In considering this question, one must remember that water follows the line of least resistance and will not enter the pervious base unless under a head—a condition so infrequent as to be negligible. As for the introduction of moisture into the subgrade by capillary action, the design provides for complete removal of free water in the trench, and the remaining moisture will probably be no more than is normally contained in the subgrade soils.

For economic considerations of backfill, etc., the accumulated water in the intercepting drain pipe is usually diverted into a main sewer when a pipe larger than 18 in. is required.

The realization of the need for pervious bases and base drainage is increasing. The proper place for a base drain is at the edge of the pavement. Also, to prevent damage to runway shoulders, the proper place to intercept surface water is at the edge of the pavement. A combined drain that performs both functions is usually more economical than a separate system. However, as in other engineering problems, the conditions must govern the decision as to whether the combined system or separate systems are to be used.

Runoff Computations.—The computation of runoff from various types of sizes of drainage areas has always been an unsatisfactory task. No two engineers working independently will arrive at the same answer. This fact is probably due more to the values selected for the variable factors than to the formula used.

Most runoff formulas contain a "runoff factor" which is expressed as a percentage of rainfall. The Horton formula (Eq. 2) applies this factor (σ) as the rainfall less detention. This seems to be a step in the right direction as it is self-evident that for like surface conditions (soil, vegetation, slope, etc.), the percentage of runoff would be greater for a 2-in. rainfall than for a 1-in. rainfall. On the other hand, the engineer is confronted with the task of assigning a value for hourly detention on which there seems to be little information.

Mr. Hathaway has done a commendable job of simplifying the Horton formula computations by the use of charts, but the solution is still more laborious than by most other formulas. However, after sifting out the essential steps, Mr. Hathaway's solution is less difficult than it appears at first.

The writer has not made a sufficient number of computations to support a definite statement on runoffs obtained by the Horton formula as compared with runoffs obtained by other formulas, such as the Burkli-Ziegler and the rational formulas. In many cases the answers are in close agreement provided the same runoff factors are used.

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(15) **"Rainfall Intensity—Frequency Data,"** by David L. Yarnell, *Miscellaneous Publication No. 204*, U.S.D.A., 1935. (*a*) Fig. 29, p. 41.

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(37) **"The Fatigue of Concrete,"** by J. L. Van Ornum, *Transactions*, Am. Soc. C. E., Vol. LVIII (1907), pp. 294–320.

(38) **"Strength of Concrete Beams,"** two theses presented jointly to the University of Missouri at Columbia, in 1906, in partial fulfilment of the requirements for the degree of Bachelor of Science in Civil Engineering, as follows: **"Comparing Various Systems of Steel Reinforcement,"** by W. K. Seitz; and **"Influence of Repeated Stresses on the Strength of Reinforced Concrete,"** by C. S. Jarvis.

(39) **"Rainfall Characteristics and Their Relation to Soils and Run-Off,"** by C. S. Jarvis, *Transactions*, Am. Soc. C. E., Vol. 95 (1931), pp. 379–458. (*a*) Fig. 4, p. 401. (*b*) Fig. 7, p. 404. (*c*) Table 5, p. 397. (*d*) Appendix I, Table 1, p. 408, et seq. (*e*) p. 398. (*f*) Fig. 5, p. 402. (*g*) Fig. 6, p. 403.

(40) **"Handbook of Applied Hydraulics,"** C. V. Davis, Editor, 1st Ed., McGraw-Hill Book Co., Inc., New York and London, 1942.

(41) **"Storm Rainfall of Eastern United States,"** *Technical Reports*, Pt. V, Miami Conservancy Dist., Dayton, Ohio, 1917.

(42) **"A Study of Excessive Rainfall,"** by H. D. Dyck and W. A. Mattice, *Monthly Weather Review*, October, 1941, p. 293.

(43) **"Primary Rôle of Meteorology in Flood Flow Estimating,"** by Merrill Bernard, *Proceedings*, Am. Soc. C. E., January, 1943, Fig. 7, p. 124.

(44) **"Comparative Studies of the Effectiveness of Different Methods for Compacting Cohesionless Soils,"** by W. Loos, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Vol. III, Cambridge, 1936.

(45) **"Investigation of Filter Requirements for Underdrains,"** U. S. Waterways Experiment Station, Vicksburg, Miss., *Technical Memorandum No. 183–1*, December 1, 1941.

(46) **"Design of Airport Runways,"** Exhibit I, Appendix B, Eng. Section Office, Chf. of Engrs., U. S. Army, January, 1941.

(47) **"Design of Airport Runways,"** Exhibit IV, Appendix B, Eng. Section Office, Chf. of Engrs., U. S. Army, January, 1941.

(48) **"Part 04—Airplane Airworthiness,"** Civil Air Regulations, Civil Aeronautics Board, Washington, D. C., as amended to August 15, 1942.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SHEAR AND BOND STRESSES IN REINFORCED CONCRETE

Discussion

BY L. E. GRINTER, AND ANDERS BULL

L. E. GRINTER,[17] M. AM. SOC. C. E.[17a]—The authors approach their problem with the logic of straightforward analysis. They accept the usual assumptions of the analysis of reinforced concrete beams and derive the standard formulas introducing such generalizations as will permit them to study the influence, on the cross section, of a thrust as well as a bending moment. Their thesis is that the thrust must be considered for the design of the cross section to resist shear and bond stresses properly.

Perhaps without intention they have chosen an example that is about the strongest reasonable case for supporting that thesis. It is a beam having a prismatic ratio (length over depth) of less than three as contrasted to a common ratio in building design of from ten to twenty. Moreover, this deep beam must resist a great thrust produced by uniform lateral pressure of the earth over the 22-ft height of the structure. Figs. 5 and 6 establish certain major discrepancies between the theoretical bond and diagonal tension stresses (not shear stress as labeled in Fig. 6) for the structure studied. Less serious discrepancies, probably of little consequence, can be anticipated for building frames, but this is no criticism of the paper, which gives only casual attention to building frames and appears intended as a study of heavy foundation structures. It is an excellent brief presentation of an interesting thesis.

The writer has ardently supported mathematical analysis as about the only device available to the designer for the production of a balanced structure; but the designer must stop and analyze the real meaning of such calculated stresses as $\frac{V}{bjd}$ to reach a logical decision as to where the refinement proposed by the authors is valuable and where it is merely theoretical "excess baggage."

NOTE.—This paper by Stanley U. Benscoter and Samuel T. Logan was published in March, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1944, by F. R. Shanley, B. J. Aleck, Dean Peabody, Jr., and William E. Wilbur.

[17] Vice-Pres. and Dean, Graduate School, Illinois Inst. of Technology, Chicago, Ill.

[17a] Received by the Secretary April 24, 1944.

A few assumptions used by the authors may be reviewed profitably because similar assumptions are always necessary for the analysis of any structure. The analysis of moments and thrusts follows from assumptions as to the loads shown as vertical uniform pressures on the top and bottom of the box (Fig. 4). Such pressure lines are never uniform but are actually curved. Likewise, the side pressures vary; they increase toward the bottom and are also likely to be unsymmetrical with respect to the vertical axis. These factors seldom can be brought into a design because they are not determined by theory but by local conditions.

Based upon conventional load assumptions, a designer computes a set of moments using stiffness factors, in all likelihood computed from I-values obtained as $\frac{b\,d^3}{12}$. If he tries the refinement of using I-values in agreement with the transformed section (see Eqs. 10), what transformed section shall he use? Evidently, since each member of a frame usually has one or more points of contraflexure, the transformed section could be either the usual one or an upside down section. In reality, over a large part of the length of the beam the effective section is certainly a section of uncracked concrete, since a crack covers only a fraction of an inch. Actually, the beam has a constantly varying moment of inertia, the degree of variation being unknown. In desperation, engineers can be excused for abandoning the idea of using the transformed section and of adopting $\frac{b\,d^3}{12}$ for the moment of inertia in determining stiffness factors; but in doing so they must not forget that this crude assumption has had its inevitable influence on computations for bending moments.

Then there is the problem of the corner action which cannot be solved by common methods of analysis. Perhaps the designer takes the theoretical span of the beam as the clear span; perhaps he chooses the span center-to-center of columns; or perhaps with corner fillets he uses less than the clear spacing between columns as the span. Whichever he chooses, the result expressed as a bending moment diagram is only an approximation—a result of judgment as well as of mathematical analysis.

Such crudities of analysis are even more obvious when the member contains both a bending moment and a thrust. The fit of a structure within confining material may shift a line of thrust by a small distance that would be of no consequence if it represented the vertical reaction under a horizontal beam; but, if it represented the location of a longitudinal thrust acting on a vertical member, it may have an enormous influence upon the stress in the member because a shift of even 1 in. in the eccentricity of a longitudinal thrust could be disastrous.

Finally, Figs. 5 and 6 reveal the theoretical fact that section moduli, fiber stresses, bending moments, bond and shearing stresses all change sharply, whereas common sense dictates that such changes are actually gradual because stress builds up slowly and not suddenly. This inaccuracy is the result of the designer's inability to treat the structure mathematically as a series of small particles. He must adopt gross procedures such as the treatment of an entire beam cross section as a single element.

It is necessary to analyze the suggestions offered in this paper with these facts in mind in order to decide just where the proposed equations should be used. The specification formula, $v = \frac{V}{b\,j\,d} < 120$ or 240, is not a statement about any stress that actually exists in the structure. It is just a rule for controlling a size of cross section, more justified by experiment and observation of its satisfaction in practical cases than by its theoretical derivation. In other words, whether the calculated shearing stress $\frac{V}{b\,j\,d}$ is high or low is immaterial if the specification writer takes care of that fact in his choice of an allowable working stress in shear.

The example offered by the authors seems to be an exaggerated case, and the effect of thrust on bond and shear in beams or columns of common length-depth ratio may not usually be of sufficient importance to consider. If most of the inaccuracies that have been mentioned in this discussion exist in the analysis of a given reinforced concrete structure, it is unlikely that the designer can hope to know the stress at a given point any closer than $\pm$ 25%. That being so, he can obtain little benefit from the theory presented in the paper unless the influence of the thrust on bond or diagonal tension proved to be of about this magnitude.

It is worth noting in Fig. 3 that the theoretical shear reaches 180 lb per sq in. on a cross section where the shear formula gives 120 lb per sq in. and the authors' theory shows the critical shear causing diagonal tension to be only 78 lb per sq in. Presumably, shear in reinforced concrete is not very serious if it cannot produce diagonal tension. However, it is well to recognize that the formulas in the paper, which show very low shear stresses, refer only to shears in the diagonal tension areas of the beam; elsewhere, larger shears than those given by $v = \frac{V}{b\,j\,d}$ may exist. Even though such increased shears may not be serious, the authors have called attention to a change in stress pattern that designers have probably been overlooking. It is a serious fault to overlook possible stress conditions. They have a habit of suddenly becoming critical. For example, the authors have not advised designers to look for the possibility that a negative thrust (a tension) can upset their computations by producing a more serious diagonal tension and bond condition than those represented by the formula $v = \frac{V}{b\,j\,d}$. Nevertheless, the theory presented by the authors can be applied to the case of negative thrust.

ANDERS BULL,[18] M. AM. SOC. C. E.[18a]—It is suggested in the paper that reinforced concrete beams may be designed more conveniently by the section modulus concept than by the methods in common use. This may be true for beams in plain bending, where the section moduli S_c and S_t are referred to the centroidal axis which, in that case, coincides with the neutral axis. When

[18] Civ. Engr., Board of Transportation, New York, N. Y.

[18a] Received by the Secretary May 19, 1944.

thrust is also acting, however, the fact that the two axes no longer coincide leads to complications.

Formulas published by the writer[19] in 1921, although similar to Eqs. 12, are applicable to both types of problems. A bending moment M and a thrust P', applied at a distance a from the center line of tensile steel, may be replaced, according to a well-known principle, by an eccentric force $P = P'$, acting at a distance $e = a + \frac{M}{P}$ from the tensile steel; that is:

$$P\,e = P\,a + M \quad \ldots\ldots (35)$$

When no thrust is acting, the term $P\,a$ drops out ($0 \times \infty$), in which case $P\,e = M$.

Treating $P\,e$ as the moment, the equilibrium equations for rotation about the respective centers of flange strain yield, for a linear stress distribution, the following formulas in which f denotes the critical stresses:

$$\frac{d}{e} = \frac{6\,p_c\,(n-1)\,(k-\alpha) - 6\,p_t\,(1-k) + 3\,k^2}{6\,p_c\,(n-1)\,(k-\alpha)\,(1-\alpha) + 3\,k^2 - k^3} \quad \ldots\ldots (36)$$

$$P\,e = b\,d^2\,\frac{6\,p_c\,(n-1)\,(k-\alpha)\,(1-\alpha) + 3\,k^2 - k^3}{6\,k}\,f_c \quad \ldots\ldots (37a)$$

$$P\,e = b\,d^2\,\frac{6\,p_c(n-1)\,(k-\alpha)\,(1-\alpha) + 3\,k^2 - k^3}{6\,n\,(1-k)}\,f_t \quad \ldots\ldots (37b)$$

Designating the fractions in Eqs. 37 by C_c and C_t, respectively:

$$f_c = \frac{P\,e}{b\,d^2\,C_c} \quad \ldots\ldots (38a)$$

and

$$f_t = \frac{P\,e}{b\,d^2\,C_t} \quad \ldots\ldots (38b)$$

Only for plain bending, when $P\,e = M$, do the denominators in Eqs. 38 represent the section moduli of the cross section about the neutral axis.

The average shearing and bond stresses are found to be:

$$v_{avg} = \frac{V}{b\,d}\left(\frac{d}{e} + \frac{p_t}{C_t}\right) \quad \ldots\ldots (39a)$$

and

$$u_{avg} = \frac{V}{b\,\Sigma_0}\left(\frac{d}{e} + \frac{p_t}{C_t}\right) \quad \ldots\ldots (39b)$$

There are numerical tables[19] from which C_c and C_t can be determined conveniently, using p_t and $\frac{d}{e}$ for arguments. When the tables are not available, the problem is solved by first determining the value of k which is the only unknown in Eqs. 37. This is easily done by means of Eq. 36, which is resolved

[19] "Computing Stresses in Reinforced Concrete Arches and Other Structures," by Anders Bull, *Concrete* (Detroit, Mich.), December, 1921, p. 242.

into the following third-degree equation:

$$\frac{d}{e}k^3 + 3\left(1 - \frac{d}{e}\right)k^2 + 6\left[p_c\,(n-1)\left(1 - \frac{d}{e} + \alpha\frac{d}{e}\right) + p_t\,n\right]k$$
$$= 6\left[p_c\,\alpha\,(n-1)\left(1 - \frac{d}{e} + \alpha\frac{d}{e}\right) + p_t\,n\right] \ldots\ldots\ldots\ldots (40)$$

For plain bending $\frac{d}{e} = 0$, and Eq. 40 is reduced to:

$$k^2 + 2\,[p_c\,(n-1) + p_t\,n]\,k = 2\,[p_c\,\alpha\,(n-1) + p_t\,n] \ldots\ldots (41)$$

Evidently the hypothetical example chosen by the authors has not been correctly treated, since the external and internal moments do not balance. For a cross section 3 in. from the face of the support (Fig. 3), the moment due to roof load is 791 in.-kips, and the moment due to thrust is $44 \times 16.2 = 713$ in.-kips—a total external moment of 1,504 in.-kips. Since the center of steel is 4.4 in. from the surface, the stresses indicated in the diagram could develop only the following internal moments:

Concrete $\ldots\ldots\ldots 12 \times 19.2 \times \frac{414}{2} \times \frac{2}{3} \times 19.2$	= 610,470
Compression steel $\ldots 0.442 \times 9 \times 414 \times \frac{14.8}{19.2} \times 14.8$	= 18,800
Tension steel $\ldots\ldots\ldots 1.203 \times 10 \times 417 \times 18.4$	= 92,310
Total internal moment (inch-pounds) $\ldots\ldots\ldots$	= 721,580

—an amount which suffices for balancing the moment due to thrust only.

Furthermore, with a thrust of 44 kips applied 3 in. above the bottom of the slab, there could be no point of inflection as in Fig. 5, the roof load moment at the center of span being $\left(78.1 - \frac{10 \times 5^2}{2}\right)12 = -563$ in.-kips on which is superimposed a thrust moment of about $44\left(\frac{42}{2} - 3\right) = +792$ in.-kips, making a total of + 229 in.-kips. The "line of thrust" shown in Fig. 5 has no physical meaning, as the thrust remains at the same height throughout the span.

For comparison, Fig. 9 shows the stresses obtained from Eqs. 36 to 40. The thrust is applied 6.7 in. from the bottom of the slab, being the resultant of $3.5 \times 4 = 14$ kips acting at the center of the cross section and $7.5 \times 4 = 30$ kips acting at the bottom.

A novel feature of the paper is the distribution of the shearing stresses across a transverse section. The findings are interesting from a theoretical point of view, but their practical value may be questioned, as the average shearing stress is so much more easily determined. The assumptions on which reinforced concrete design is based are far from consistent in regard to tensile stresses in concrete. Such stresses are considered when diagonal but are disregarded when longitudinal, although the latter stresses obviously contribute

to the bending strength. The aim is to simplify the computations. For the same reason any highly refined treatment would seem somewhat futile. After all, the allowable stresses governing the design are determined by the cut-and-try method. The possibility of future excavation alongside a subway structure like the one in Fig. 4 must also be considered as it may cause the thrust to be removed temporarily, eliminating its relieving effect on the shearing stresses.

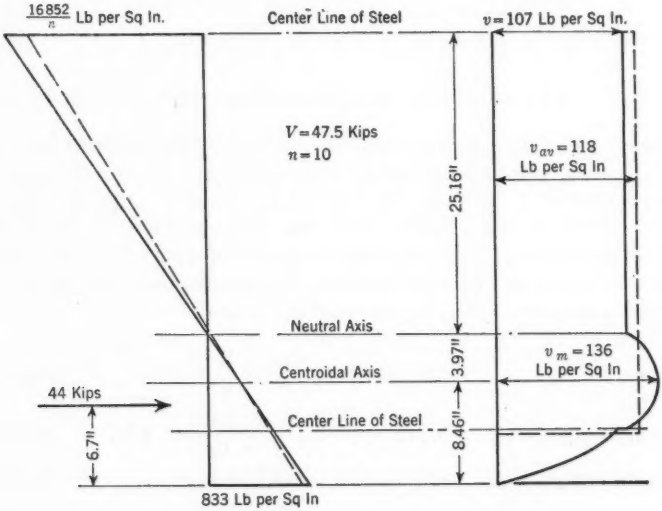


FIG. 9

The significance of the bulging shape of the shear distribution curve is easily seen by investigating another cross section, located a little closer to the center of span. The new stress distribution line (inclined dashed line in Fig. 9) is then found to intersect the old line at the centroidal axis, showing that the compressive stresses below the axis are reduced whereas those above are increased. These changes conform to the shape of the curve, whose first derivative $\frac{dx}{dy}$ is proportional to the change in compressive stress at any given height y.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ANALYSIS OF RIGID FRAMES BY SUPERPOSITION

Discussion

BY A. W. FISCHER, LEON BLOG, ARTHUR B. MCGEE, AND JAROSLAV POLIVKA

A. W. FISCHER,[7] ESQ.[7a]—The analysis of rigid frames by superposition as presented by the author is certainly another valuable method for the analysis of rigid frames. The illustrative examples presented in the paper show that

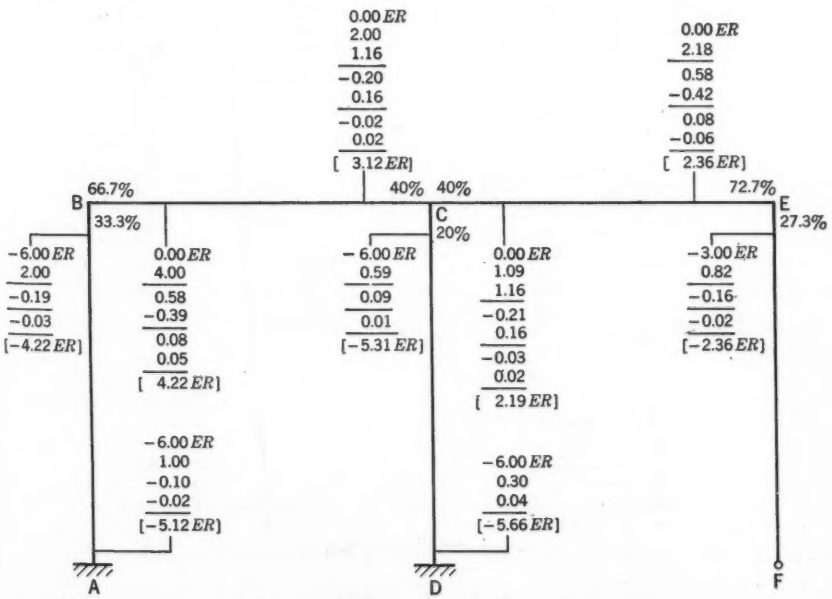


FIG. 14.—UNSYMMETRICAL THREE-COLUMN BENT

NOTE.—This paper by David M. Wilson was published in February, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1944, by F. S. Merritt, Ralph W. Stewart, and John E. Goldberg.

[7] Engr. (Naval Architect), Navy Dept., Washington, D. C.

[7a] Received by the Secretary April 17, 1944.

the method is general and can be used for complex frames. In 1938 (20)[7b] the writer found the method of superposition very practicable for calculating moments in structures several bays wide and several stories high.

Example 2. Unsymmetrical Three-Column Bent.—The method of moment distribution as presented by Professor Wilson is general. In some cases considerable time can be saved by taking advantage of permissible short cuts. For example, the moment distribution in Fig. 3(*b*) can be shortened considerably by first balancing a joint and then carrying over one half of the balancing moment to the adjacent joints, before the adjacent joints are balanced. This method is correct theoretically, and the writer uses it in all his moment distribution analyses. Fig. 14 shows this method of balancing moments beginning at joint B. After carrying over one half of the balancing moment to the adjacent joints, the next step is to balance joint E. Finally, carry over one half of the balancing moment to joints adjacent to joint E and then balance joint C.

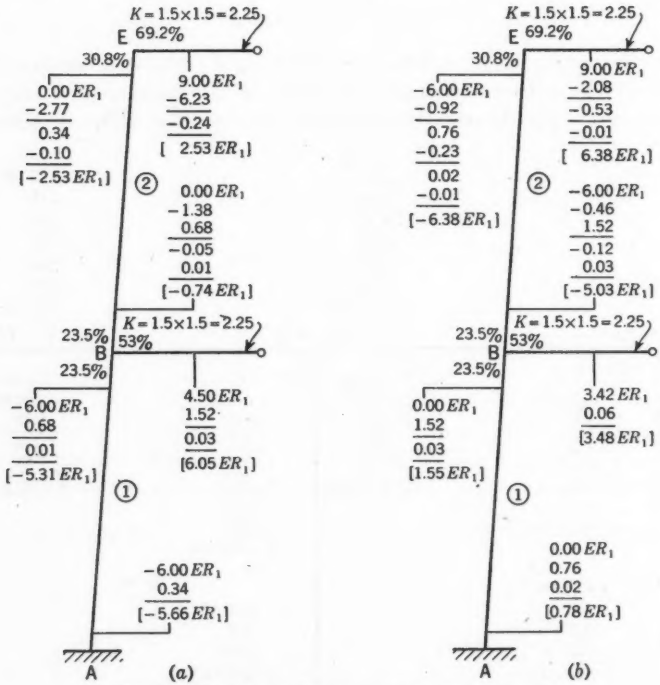


FIG. 15.—TWO-STORY BENT WITH HORIZONTAL LOADS

On comparison with the formal procedure of the paper the amount of labor is found to be reduced considerably.

Example 7. Two-Story Bent with Horizontal Loads.—For a symmetrical structure, such as Fig. 9, the *K*-factors for the horizontal members can be

[7b] Numerals in parentheses, thus: (20), refer to corresponding items in the Bibliography (see Appendix in the paper), and at the end of discussion in this issue.

modified slightly by assuming that there is a point of inflection, or hinge, at the center, which makes the K-factor equal to $\frac{I}{0.5\,L} \times \frac{3}{4} = 1.5\,\frac{I}{L}$. Thus, the K-factors for members BC and EF are equal to $1.5 \times 1.5 = 2.25$, and no balancing moment need be carried over to the adjacent joints for these members. Fig. 15(*a*) gives the moment distribution for the final moments shown in Fig. 9(*b*), and Fig. 15(*b*) shows the moment distribution for the final moments shown in Fig. 9(*c*). With this slight modification, considerable labor can be saved and the method is theoretically correct.

Example 6. Trapezoidal Bent.—Fig. 16(*a*) represents a single-story trapezoidal bent similar to Fig. 8, but with a bottom member and a horizontal load of 1,000 lb applied at joint A (compare analysis of this bent by A. Amirikian, M. Am. Soc. C. E. (1*a*)). For this example, $E = 1$, $R_2 = R_1$, and $R_3 = -R_1$. Since K for member AD is equal to 2 (Fig. 16), the fixed-end moment $M_{AD} = M_{DA} = -2 \times 6\,R_1 = -12\,R_1$. The fixed-end moment $M_{AB} = M_{BA} = 2 \times 6\,R_1 = 12\,R_1$. The fixed-end moment $M_{BC} = M_{CB} = -3 \times 6\,R_1 = -18\,R_1$ (Fig. 16(*b*)). The first joint balanced is joint C and one half of the balanced moments are carried to the adjacent joints. Next balance joint D and carry one half of the balanced moments to its adjacent joints, balance joint B and carry one half of the balanced moments to its adjacent joints, and balance joint A, carrying one half of the balanced moments to its adjacent joints. Repeat the process in the order of joints C, D, B, and A. The final moments are shown in brackets in Fig. 16(*b*), thus accounting for steps 1, 2, and 3.

By step 4 in the author's procedure (see Fig. 16(*c*)):

$$H_1 + H_2 = 1{,}000 \qquad (37)$$

also, in Fig. 16(*d*),

$$V_1 = 4.085\,R_1 \qquad (38a)$$

and

$$V_2 = 4.085\,R_1 \qquad (38b)$$

The values of V_1 and V_2 in Eqs. 38 are shown in Figs. 16(*e*) and 16(*f*), respectively.[7c] In Fig. 16(*e*),

$$H_1 = \frac{29.51\,R_1}{12} \qquad (39a)$$

and, in Fig. 16(*f*),

$$H_2 = \frac{40.11\,R_1}{12} \qquad (39b)$$

Substituting Eqs. 39 in Eq. 37 and solving, $R_1 = \frac{12{,}000}{69.62}$.

In step 5, substituting the value of R_1 from step 4 in Fig. 16(*b*) gives the following final moments in foot-pounds: $M_{AB} = 2{,}006 = -M_{AD}$; $M_{BA} = 2{,}218 = -M_{BC}$; $M_{DA} = -1{,}672 = -M_{DC}$; and $M_{CB} = -1{,}879 = -M_{CD}$.

Example 5A. Vierendeel Truss with Inclined Chords.—This problem, of course, differs somewhat from Example 5 of the paper (compare analysis by Mr. Amirikian (1*b*)). Fig. 17(*a*) represents a five-bay Vierendeel truss with a 200-lb load at each lower panel point. The value of E is considered as unity

[7c] Correction for *Transactions:* In Fig. 16, indicate final sketch as "(*f*)."

throughout, and the values of $I/L = K$ are as shown. Figs. 17(*b*) and 17(*c*) show the left portion of the truss in the distorted condition with the R-values expressed in terms of R_1 and R_2 using Eq. 21. For this particular symmetrical

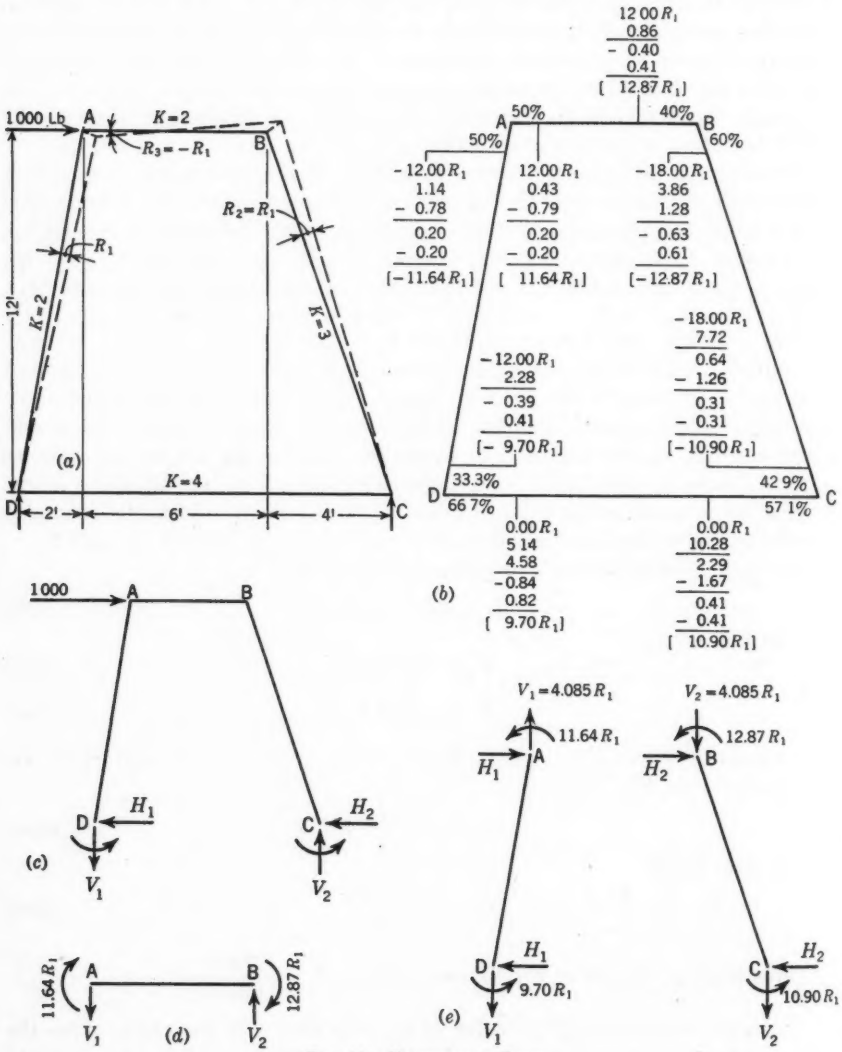


FIG. 16.—TRAPEZOIDAL BENT

truss assume a hinge in the center of the members AA′, BB′, and CC′ and multiply each K-value by 1.5 so that, with this slight modification, no balancing moment is required to be carried over to the adjacent joint for these members.

The fixed-end moments 6 $K R_1$ for the various members are shown in Fig. 18(*a*). For the moment distribution method the sequence is: Joints A, C,

and B. For member CD the carry-over balanced moment from joint D to joint C is one half of the balance moment at joint C with the sign changed, because then it is necessary only to apply the moment distribution method to the left part of the truss when the load and truss are symmetrical. The final moments in terms of R_1 are shown in brackets.

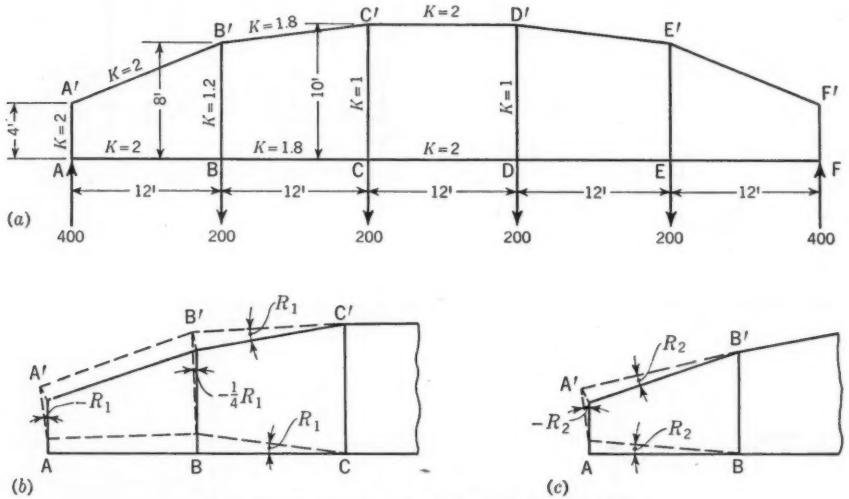


FIG. 17.—VIERENDEEL TRUSS WITH INCLINED CHORDS

Fig. 18(*b*) gives the fixed-end moments in terms of R_2 and the sequence of balancing is: Joints B, A, and C, thus accounting for steps 1, 2, and 3.

In Fig. 18(*c*), for step 4:

$$H_1 + H_2 = 200 \quad \ldots\ldots (40a)$$

and, in Fig. 18(*d*),

$$H_3 + H_4 = 400 \quad \ldots\ldots (40b)$$

In Fig. 18(*c*), with $\Sigma M = 0$,

$$V_1 = \frac{-10.72\,R_1 + 2.40\,R_2 + 7{,}200}{10} \quad \ldots\ldots (41a)$$

and, in Fig. 18(*d*), from summation of moments = 0,

$$V_2 = \frac{1.54\,R_1 - 15.72\,R_2 + 4{,}800}{8} \quad \ldots\ldots (41b)$$

The values of V_1 and V_2 in Eqs. 41 are shown acting in Figs. 18(*e*), 18(*f*), 18(*g*), and 18(*h*). In Figs. 18(*e*) and 18(*f*), respectively, with $\Sigma M = 0$, and Figs. 18(*g*) and 18(*h*),

$$H_2 = \frac{44.68\,R_1 - 22.25\,R_2 + 7{,}200}{60} \quad \ldots\ldots (42a)$$

$$H_1 = \frac{11.08\,R_1 - 4.93\,R_2}{12} \quad \ldots\ldots (42b)$$

$$H_4 = \frac{7.50\,R_1 + 21.26\,R_2 + 4{,}800}{24} \quad \ldots\ldots (42c)$$

FIG. 18.—MOMENT DISTRIBUTION, VIERENDEEL TRUSS

and

$$H_3 = \frac{2.98\,R_1 + 18.49\,R_2}{12} \ldots\ldots\ldots\ldots\ldots\ldots (42d)$$

Substituting Eqs. 42 in Eqs. 40,

$$100.08\,R_1 - 46.90\,R_2 = 4{,}800 \ldots\ldots\ldots\ldots\ldots\ldots (43a)$$

and

$$13.46\,R_1 + 58.24\,R_2 = 4{,}800 \ldots\ldots\ldots\ldots\ldots\ldots (43b)$$

Solving Eqs. 43 by determinants, $R_1 = 78.12$ and $R_2 = 64.36$.

The numerical values of the moments in Figs. 18(*a*) and 18(*b*) may now be calculated (step 5), and the following moments in foot-pounds at the ends of all members are determined by summation: $M_{AB} = M_{A'B'} = -977$; $M_{BA} = M_{B'A'} = -446$; $M_{BC} = M_{B'C'} = -207$; $M_{CB} = M_{C'B'} = -342$; $M_{CD} = M_{C'D'} = 136$.

The foregoing is intended to lend emphasis to the writer's sincere opinion that the method of superposition as presented by Professor Wilson is an unusually valuable tool for the solution of rigid frames.

LEON BLOG,[8] ESQ.[8a]—The importance of using the method of superposition when analyzing rigid frames is emphasized in this paper; but the paper has additional value. In the several examples, the author has explained various simplifications and clarifications previously unknown to the writer. Although superposition was not required for solving Example 1, there is considerable merit in the technique of its solution. By separating the R-term in Eqs. 1 and by writing it separately as in Eqs. 2, and then by expressing the R-term of other members in terms of that of one member, a considerable simplification in the statement of the values for the end moments occurs. Through stating the values of the angles of twist, θ, in terms of $E\,R$ as in Eqs. 6, it becomes unnecessary to evaluate θ and R quantitatively. Values of θ and R in radians are generally of a low order and lead to errors in the final moment values unless carried throughout the solution with many significant figures. Another simplification is that the value E occurs in both sides of Eqs. 6, so that, by the author's method, it need not be considered separately. Only the product $E\,R$ must be evaluated and substituted in the values of the end moments which are expressed solely in terms of $E\,R$. Use of the value of $E\,R$ leads to a minimum of error in the moment values sought. The technique just described is better than any known to the writer for the solution of rigid frames by either the slope deflection or the moment distribution methods.

Superposition is a method of solving, by steps, certain problems to which it is applicable. When solving problems dealing with rigid frames involving asymmetry of any kind, one-step solutions by moment distribution are not possible. Two steps are required: (*a*) Moment distribution to evaluate moments due to pure rotation of the joints; and (*b*) moment distribution to obtain moments due to movements occurring transversely to the axis of each member that can be deflected. The moments evaluated in each step are added alge-

[8] Structural Engr., City of Los Angeles, Los Angeles, Calif.

[8a] Received by the Secretary April 21, 1944.

FIG. 18.—MOMENT DISTRIBUTION, VIERENDEEL TRUSS

braically at each joint to derive the required moment. Step 1 may be solved by any good system of moment distribution. Step 2 is often solved by the unbalanced shear method (2). Hardy Cross, M. Am. Soc. C. E., does not call step 2 anything, but unbalanced shears are actually the tools required when correcting for sidesway by his method.

Professor Wilson does not use the unbalanced shear method in step 2. In solving Example 2, Fig. 3(*b*), he distributes fixed-end moments which are expressed in terms of $E\,R$. Then he evaluates $E\,R$ and uses it to evaluate the distributed moments. The moments thus found are added algebraically to those in Fig. 3(*a*) to get the working moments. This consummates a very direct solution of Example 2.

Example 3 is an extension of the technique exhibited by Example 2. In analyzing two stories, it is necessary to use a set of R-values: R_1 appertains to the lower story and R_2 to the upper tier. After that variation from Example 2, the technique of solution parallels that for Example 2.

For the solution of Example 4, the author evaluates moments in terms of $E\,\Delta$. Here, inspection will not yield R-values for the vertical members in terms of the R-value of a selected vertical member as illustrated in Example 1. In this example it was evident that the R-values for vertical members joined by the same horizontal member are inversely proportional to the length of the vertical members. In Fig. 5(*a*), the joint at D between points C and E precludes evaluation of deflection angles by inspection. Otherwise the method of solution closely follows that for Examples 2 and 3.

For solving the Vierendeel truss of Example 5, Professor Wilson is able to profit by expressing moments in terms of $E\,R$. Otherwise, the solution is self-explanatory.

Equally valuable is the exposition of the relation between the values of R for the members of a quadrangle. The knowledge of this relation permits a very direct attack upon the type of problem considered in the ensuing comment. The meaning of R_3 in Fig. 7 may be made somewhat clearer by drawing a horizontal line through B' toward the right to meet a perpendicular drawn downward from C'. Then[8b]

$$R_3 = \frac{v_C + v_B}{l_3} \qquad (44)$$

For the solution of the frame of Example 6, the R-relation in Eqs. 20 and 21 was used. Fig. 8 is not a quadrangle, but the fixity of the legs AB and CD is analogous to having a member such as AD of Fig. 7. Therefore, Eqs. 20 and 21 are applicable to the solution of Example 6. The distributed moments are determined by using $E\,R_1$ in a manner closely resembling the solution of Example 2.

Example 7, Fig. 9, is solved easily by using the quadrangle concept. The lower story has legs which are fixed, and so a fictitious member AD may be assumed. The second story is a full quadrangle. Otherwise the solution closely follows that of Example 3.

[8b] Correction for *Transactions:* In Fig. 7 change "V_B" and "V_C" to "v_B" and "v_C," respectively.

In Fig. 10(*b*) of Example 8, the author uses the quadrangle. The dashed line EB corresponds to AD of the basic quadrangle. Moments are all expressed in terms of $E\,R_1$ and $E\,R_2$, and thereafter the equations of statics dominate the solution.

Finally, the writer believes that Professor Wilson has made a distinct contribution to the analysis of rigid frames of homogeneous material. Should any of the members be variable in shape, the resultant variation of end moment of inertias would preclude the simplification which results in Eqs. 4 by using the concept of relative K-values. However, each Eq. 4 would merely contain the K-value for that member and would lead to some additional numerical work. If the frame is not homogeneous as to material, modular ratios such as those for reinforced concrete may be used to establish all the equations in terms of the E of a selected member. This also would involve some extra numerical work, but the extra work would be common to other known methods.

ARTHUR B. MCGEE,[9] ESQ.[9a]—The principle of superposition is of great value for the analysis of rigid frames. The examples in the paper illustrate the method of procedure in a workable manner; but, since Examples 1, 2, 3, 4, 6, and 8 can be solved by shorter and easier methods, they do not demonstrate the value of the method. To do the method justice and to show its great value in solving difficult problems the writer wishes to show its application to a two-bay, saw-toothed structure with tapering members.

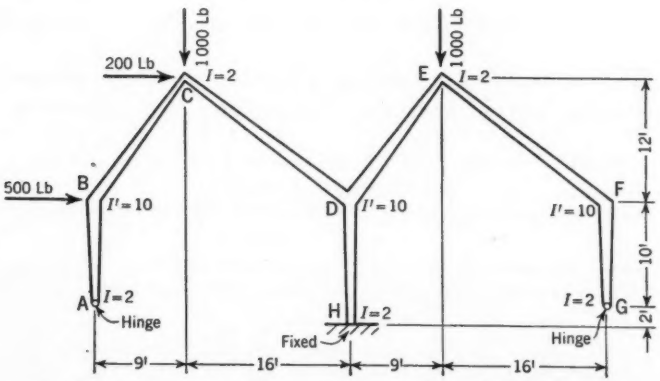


FIG. 19.—TWO-BAY, SAW-TOOTHED STRUCTURE WITH TAPERING MEMBERS

Fig. 19 shows such a frame loaded unsymmetrically with vertical and horizontal loads. The tapering members have been given the same moments of inertia at the ends so that one set of beam constants will suffice for all members. These constants are taken from the tables compiled by Walter Ruppel, Assoc. M. Am. Soc. C. E. (21). The application of these constants to the several members is given in Fig. 20 (22).

[9] Structural Design Engr., Bureau of Eng., City of Los Angeles, Los Angeles, Calif.

[9a] Received by the Secretary April 28, 1944.

The notation used in this discussion varies slightly from the author's but should cause no confusion. The symbol d is used herein for the linear displacement of one end of a member, and Δ represents an angle equal to an $\frac{M}{E\,I}$-area. Also the writer considers an end moment as the moment in a member rather than a resisting moment acting on the member. This, of

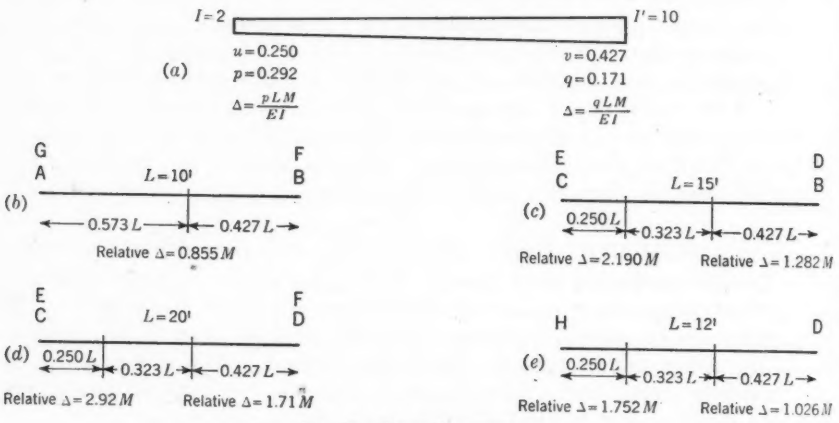


FIG. 20.—BEAM COEFFICIENTS

course, reverses the signs of the end moments from those used by Professor Wilson.

Table 1 gives the effect of applying a positive (clockwise) moment to each joint of the frame, without sidesway. The table may be computed by moment

TABLE 1.—EFFECTS OF MOMENTS APPLIED TO JOINTS
(Not Corrected for Sidesway)

+1,000 ft-lb moment applied at joint:	MEMBER											
	BA	BC	CB	CD	DC	DH	DE	ED	EF	FE	FG	HD
B	−569	−431	− 69	+ 69	+ 41	− 24	− 17	− 3	+ 3	+ 1	− 1	− 8
C	+249	−249	−558	−442	−260	+155	+105	+ 17	− 17	− 8	+ 8	+ 52
D	− 22	+ 22	+ 49	− 49	−240	−452	−308	− 49	+ 49	+ 24	− 24	−151
E	+ 10	− 10	− 22	+ 22	+107	+201	−308	−575	−425	−211	+211	+ 67
F	− 1	+ 1	+ 3	− 3	− 14	− 27	+ 41	+ 77	− 77	−371	−629	− 9

distribution. The writer, however, prefers the traverse method (22). For a moment at joint B the traverse is shown in Fig. 21. The moments on the diagram are multiplied by $\frac{1{,}000}{903.97}$ to obtain the results for joint B in Table 1. A corresponding traverse must be run for joint F. The intermediate joints, C, D, and E are easily calculated by using data taken from the two traverses already run. Table 1 will apply to any combination of fixed-end moments that may act on the frame.

When the frame is permitted to deflect as indicated in Fig. 22(*a*), the relative displacements of the members AB, BC, and CD are shown in Table 2. In the triangle abc, side ab is drawn normal to member AB, side bc normal to member BC, and, likewise, side ca normal to member CD. These displace-

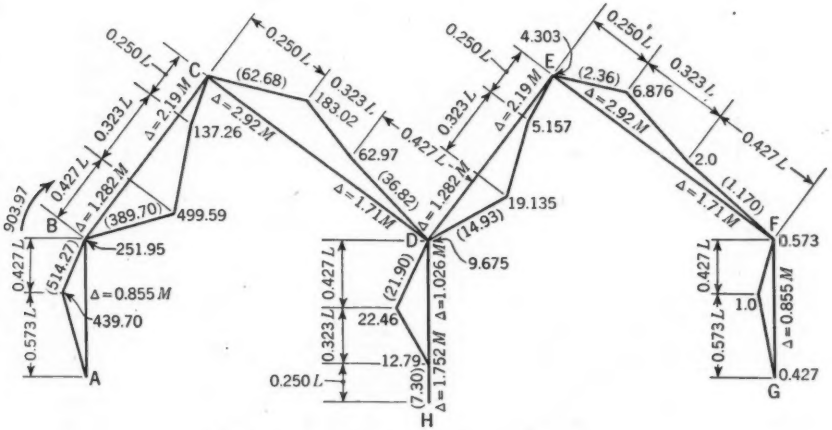


FIG. 21.—SOLUTION BY TRAVERSE METHOD

ments are d, $-0.8\,d$, and $0.6\,d$, respectively. The equations for fixed-end moments, corresponding to $M'_{AB} = M'_{BA} = 6\,E\,K\,\frac{d}{L}$ for uniform moment of inertia members, are as follows:

$$M_L = \frac{E\,I\,d}{(1-u-v)\,p\,L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (45a)$$

and

$$M_R = \frac{E\,I\,d}{(1-u-v)\,q\,L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (45b)$$

Table 2 shows the fixed-end moments for members AB, BC, CD, and DH, Fig. 19. Member DE is like member BC, member EF is like member CD, and member FG is like member BA.

There are three conditions of deformation for this frame. The first condition is shown in Fig. 22(*a*). The coefficients of $E\,d$, as calculated in Table 2, have been multiplied by 1,000 arbitrarily to place the decimal point in a more convenient location. The fixed-end moment coefficients of $\frac{E\,d_1}{1{,}000}$ for members AB, BC, and CD are placed on the diagram in their respective locations and the moments are balanced by the use of Table 1. Thus, at joint B, $204.12 - 128.75 = 75.37$, a clockwise moment which must be balanced throughout the frame. Therefore, $\frac{75.37}{1{,}000}$ times the values in Table 1 for joint B will be added algebraically to the respective end moments of the members. Likewise, at joint C, $-75.40 + 31.81 = -43.59$, the unbalanced moment, and

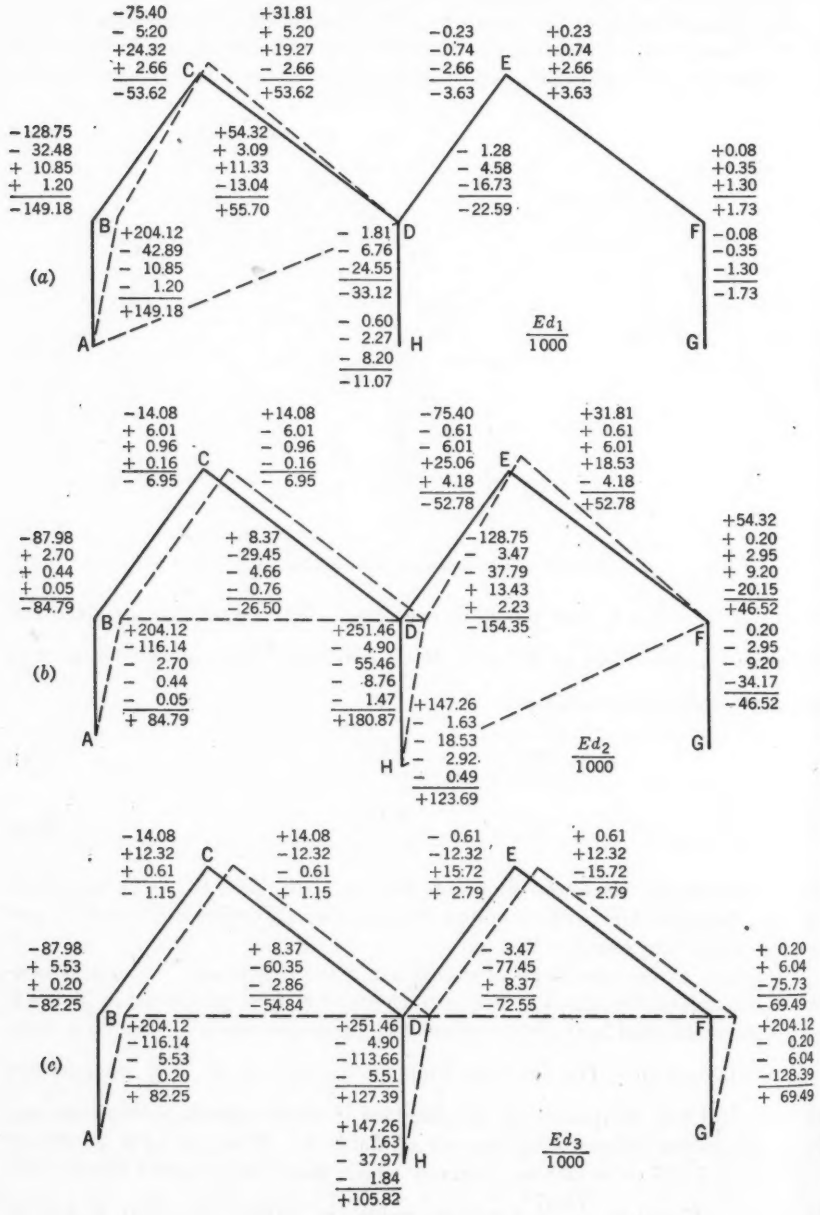
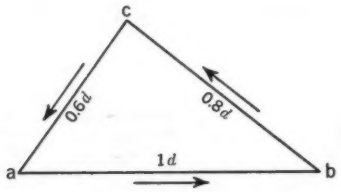
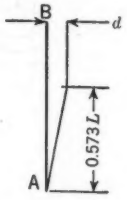
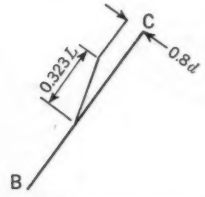
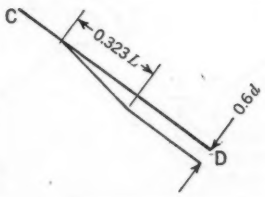
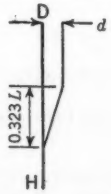

−75.40
− 5.20
+24.32
+ 2.66
−53.62
+31.81
+ 5.20
+19.27
− 2.66
+53.62
−0.23
−0.74
−2.66
−3.63
+0.23
+0.74
+2.66
+3.63
C
E
−128.75
− 32.48
+ 10.85
+ 1.20
−149.18
+54.32
+ 3.09
+11.33
−13.04
+55.70
− 1.28
− 4.58
−16.73
−22.59
+0.08
+0.35
+1.30
+1.73
B
D
F
(a)
+204.12
− 42.89
− 10.85
− 1.20
+149.18
− 1.81
− 6.76
−24.55
−33.12
− 0.08
− 0.35
− 1.30
− 1.73
− 0.60
− 2.27
− 8.20
−11.07
A
H
G
$\frac{Ed_1}{1000}$
−14.08
+ 6.01
+ 0.96
± 0.16
− 6.95
+14.08
− 6.01
− 0.96
− 0.16
− 6.95
−75.40
− 0.61
− 6.01
+25.06
+ 4.18
−52.78
+31.81
+ 0.61
+ 6.01
+18.53
− 4.18
+52.78
C
E
−87.98
+ 2.70
+ 0.44
± 0.05
−84.79
+ 8.37
−29.45
− 4.66
− 0.76
−26.50
−128.75
− 3.47
− 37.79
+ 13.43
+ 2.23
−154.35
+54.32
+ 0.20
+ 2.95
+ 9.20
−20.15
+46.52
B
D
F
(b)
+204.12
−116.14
− 2.70
− 0.44
− 0.05
+ 84.79
+251.46
− 4.90
− 55.46
− 8.76
− 1.47
+180.87
− 0.20
− 2.95
− 9.20
−34.17
−46.52
+147.26
− 1.63
− 18.53
− 2.92
− 0.49
+123.69
A
H
G
$\frac{Ed_2}{1000}$
−14.08
+12.32
+ 0.61
− 1.15
+14.08
−12.32
− 0.61
+ 1.15
− 0.61
−12.32
+15.72
+ 2.79
+ 0.61
+12.32
−15.72
− 2.79
C
E
−87.98
+ 5.53
+ 0.20
−82.25
+ 8.37
−60.35
− 2.86
−54.84
− 3.47
−77.45
+ 8.37
−72.55
+ 0.20
+ 6.04
−75.73
−69.49
B
D
F
(c)
+204.12
−116.14
− 5.53
− 0.20
+ 82.25
+251.46
− 4.90
−113.66
− 5.51
+127.39
+204.12
− 0.20
− 6.04
−128.39
+ 69.49
+147.26
− 1.63
− 37.97
− 1.84
+105.82
A
H
G
$\frac{Ed_3}{1000}$

FIG. 22

TABLE 2.—RELATIVE DISPLACEMENTS AND MOMENTS, MEMBERS AB, BC, CD, AND DH, FIG. 19



Member	Displacements	Moments
AB		$M_{BA} = \frac{d \times E \times 2}{0.573 \times 0.171 \times 100} = 0.20412\ E\ d$
BC		$M_{CB} = \frac{-0.8\ d \times E \times 2}{0.323 \times 0.292 \times 225} = -0.07540\ E\ d$ $M_{BC} = \frac{-0.8\ d \times E \times 2}{0.323 \times 0.171 \times 225} = -0.12875\ E\ d$
CD		$M_{CD} = \frac{0.6\ d \times E \times 2}{0.323 \times 0.292 \times 400} = 0.03181\ E\ d$ $M_{DC} = \frac{0.6\ d \times E \times 2}{0.323 \times 0.171 \times 400} = 0.05432\ E\ d$
DH		$M_{DH} = \frac{d \times E \times 2}{0.323 \times 0.171 \times 144} = 0.25146\ E\ d$ $M_{HD} = \frac{d \times E \times 2}{0.323 \times 0.292 \times 144} = 0.14726\ E\ d$

$\frac{-43.59}{1{,}000}$ times the moments shown for a +1,000 ft-lb moment at joint C in Table 1 will spread this unbalanced moment throughout the frame. The same procedure is followed for the unbalanced +54.32 moment at joint D. The end moments for each member are now added and the first condition of deformation is finished.

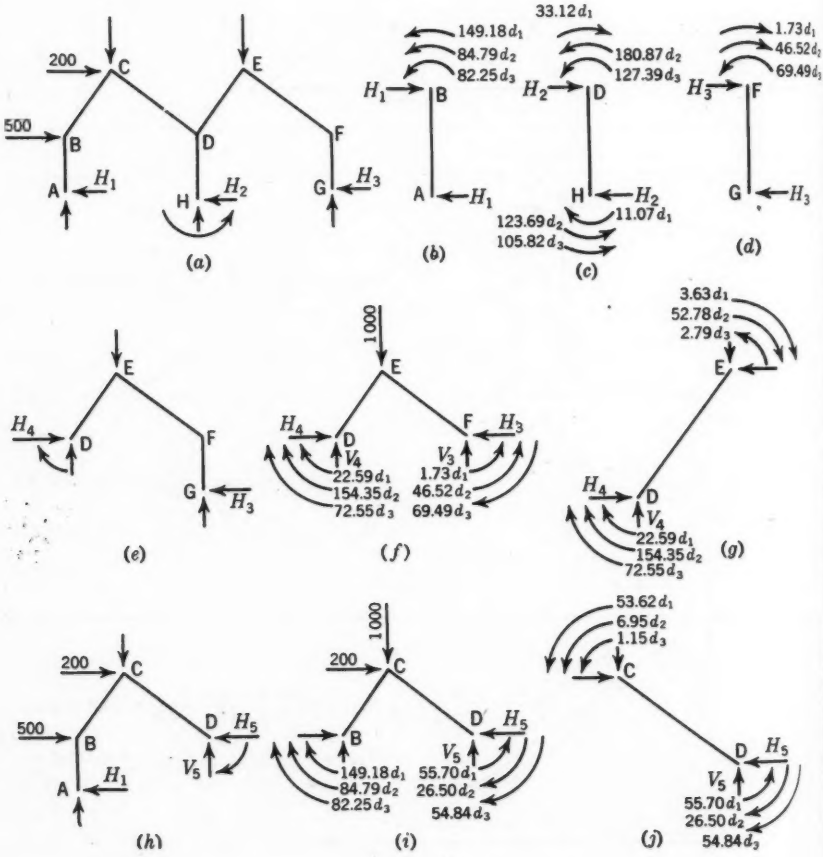


FIG. 23

Fig. 22(*b*) shows the second condition of deformation, and the coefficients of $\frac{E\,d_2}{1{,}000}$ are computed as before. The third condition is solved in Fig. 22(*c*).

In Fig. 23 and the remainder of this discussion, the expressions $\frac{E\,d_1}{1{,}000}$, $\frac{E\,d_2}{1{,}000}$, and $\frac{E\,d_3}{1{,}000}$ have been contracted to d_1, d_2, and d_3, respectively.

From Fig. 23(*a*) it is evident that

$$H_1 + H_2 + H_3 = 700 \quad \ldots\ldots\ldots\ldots\ldots\ldots (46a)$$

In Fig. 23(*b*), from $\Sigma M = 0$,

$$H_1 = 14.918\, d_1 + 8.479\, d_2 + 8.225\, d_3 \quad \ldots\ldots\ldots\ldots (46b)$$

Similarly, in Fig. 23(*c*),

$$H_2 = -3.682\, d_1 + 25.38\, d_2 + 19.434\, d_3 \quad \ldots\ldots\ldots\ldots (46c)$$

and, in Fig. 23(*d*),

$$H_3 = -0.173\, d_1 - 4.652\, d_2 + 6.949\, d_3 \quad \ldots\ldots\ldots\ldots (46d)$$

Substituting Eqs. 46*b*, 46*c*, and 46*d* in Eq. 46*a* gives

$$11.063\, d_1 + 29.207\, d_2 + 34.608\, d_3 = 700 \quad \ldots\ldots\ldots\ldots (46e)$$

From Fig. 23(*e*),

$$H_4 - H_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (47a)$$

In Fig. 23(*f*), taking moments about point F,

$$V_4 = 600 - 0.834\, d_1 - 4.313\, d_2 - 5.682\, d_3 \quad \ldots\ldots\ldots\ldots (47b)$$

In Fig. 23(*g*), taking moments about point E and substituting the value of V_4 from Eq. 47*b*,

$$H_4 = 480 + 1.560\, d_1 + 14.026\, d_2 + 1.552\, d_3 \quad \ldots\ldots\ldots\ldots (47c)$$

Substituting Eqs. 46*d* and 47*c* in Eq. 47*a* gives

$$480 + 1.733\, d_1 + 18.678\, d_2 - 5.397\, d_3 = 0 \quad \ldots\ldots\ldots\ldots (47d)$$

From Fig. 23(*h*),

$$H_1 + H_5 = 700 \quad \ldots\ldots\ldots\ldots\ldots\ldots (48a)$$

In Fig. 23(*i*), taking moments about point B,

$$V_5 = 456 + 3.739\, d_1 + 4.452\, d_2 + 5.484\, d_3 \quad \ldots\ldots\ldots\ldots (48b)$$

In Fig. 23(*j*), taking moments about point C and substituting the value of V_5 from Eq. 48*b* gives the equation,

$$H_5 = 608 + 14.095\, d_1 + 4.307\, d_2 + 2.838\, d_3 \quad \ldots\ldots\ldots\ldots (48c)$$

Substituting Eqs. 46*b* and 48*c* in Eq. 48*a* gives

$$29.013\, d_1 + 12.786\, d_2 + 11.063\, d_3 = 92 \quad \ldots\ldots\ldots\ldots (48d)$$

Solving Eqs. 46*e*, 47*d*, and 48*d* simultaneously gives $d_1 = -3.062$, $d_2 = -15.512$, and $d_3 = 34.280$.

The moments in Figs. 22(*a*), 22(*b*), and 22(*c*) multiplied by their respective *d*-values are shown in Table 3. The final values of the moments at the ends of all members are given in the bottom line of Table 3.

Additional bays can be added to the frame in Fig. 19 without increasing the problem unduly. Each additional bay would add one more condition of deformation. In other words, such a three-bay structure would require four conditions of deformation; a four-bay structure would call for five; etc.

TABLE 3.—END MOMENTS IN FOOT-POUNDS

Factors	MEMBER											
	BA	BC	CB	CD	DC	DH	DE	ED	EF	FE	FG	HD
$d_1 = -3.062$	− 457	+ 457	+164	−164	− 170	+ 102	+ 69	+ 11	− 11	− 5	+ 5	+ 34
$d_2 = -15.512$	−1,315	+1,315	+108	−108	+ 411	−2,806	+2,394	+819	−819	− 722	+ 722	−1,918
$d_3 = 34.280$	+2,819	−2,819	− 39	+ 39	−1,880	+4,367	−2,487	+ 96	− 96	−2,382	+2,382	+3,627
Total....	+1,047	−1,047	+233	−233	−1,639	+1,663	− 24	+926	−926	−3,109	+3,109	+1,743

The solution of this type of problem by superposition may be rather long, but the individual steps are logical and easily followed and in the end a difficult problem has been solved.

JAROSLAV POLIVKA,[10] M. AM. SOC. C. E.[10a]—The principle of superposition is generally recommended and used in cases in which the direct analysis either involves the solution of a greater number of simultaneous equations or requires a relatively tedious determination of certain constants of the structure. For example, using the slope-deflection method, the examples presented in the paper require the solution of the following number of simultaneous equations:

Example No.	Number of equations	Example No.	Number of equations
1	3	5	8
2	4	6	3
3	4	7	4
4	6	8	5

In Examples 3 and 7, there are only four equations, of course, since the frames are symmetrical.

In most cases several load systems must be investigated, and for this reason preference should be given to a method of analysis by which certain constant values or characteristics of the given structure, common and applicable to any type of loading, can be determined. The method of analysis presented by the author possesses this advantage, since the computation of the internal forces (bending moments, shears, and thrusts) produced by lateral displacements of the joints is treated independently of transverse load.

Since an accuracy greater than 4% is usually not required for design purposes (except for checking the more complicated statically indeterminate structures), graphical or combined methods are useful for types of framed structures to which the method of superposition is applicable. One of the most promising methods is that of fixed (characteristic) points, which has not yet been fully utilized in the United States. The solution is general and can be applied to

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[10a] Received by the Secretary May 4, 1944.

hinged and elastically yielding supports. The latter condition (yielding supports) is often more serious in engineering practice than either of the extreme conditions of fully restrained or hinged supports.

The designer can adapt graphical, semigraphical, or algebraic processes to his purpose. Graphical devices are quicker; they are easier to check; and they are as accurate as the precision of the draftsman who prepares them.

A graphical solution by fixed points, applied to Example 2, is demonstrated by reference to Fig. 24. The determination of the fixed (characteristic) points, U and V, of each structural member is shown in Fig. 24(*a*) using the values of the elastic weights, $G = \frac{L}{E\,I} = \frac{1}{E\,K}$, and the angular displacements, δ of the joints. The distances of the characteristic points, u and v, can be checked or calculated by the following relationships: $u_1 = \frac{L_1}{3}$ (full restraint at point A); $u_2 = \frac{L_2}{3}\,\frac{G_2}{G_2 + 2\,\delta_{BA}}$; and $u_4 = \frac{L_4}{3}\,\frac{G_4}{G_4 + 2\,\delta_{CAD}}$, in which $\delta_{CAD} = \frac{\delta_{CA}\,\delta_{CD}}{\delta_{CA} + \delta_{CD}}$, etc. The angular displacement δ_{BA} has the significance of the elastic weight of support of the member BC at point B and equals $\frac{G_1}{4}$, because of restraint at point A. The successive angular displacements can be expressed in terms of distances between the opposite fixed points and the supports; for example, $\delta_{CA} = \frac{G_2}{6}\left(3 - \frac{L_2}{L_2 - u_2}\right)$, etc.

Fig. 24(*b*) shows the bending moments of the bent, subjected to a central load on member BC, without considering the sidesway, found by the fixed points and the crosslines for the loaded span. The crosslines are determined by values K_B and K_C, derived from fixed-end moments or from formulas for each type of transverse loading.

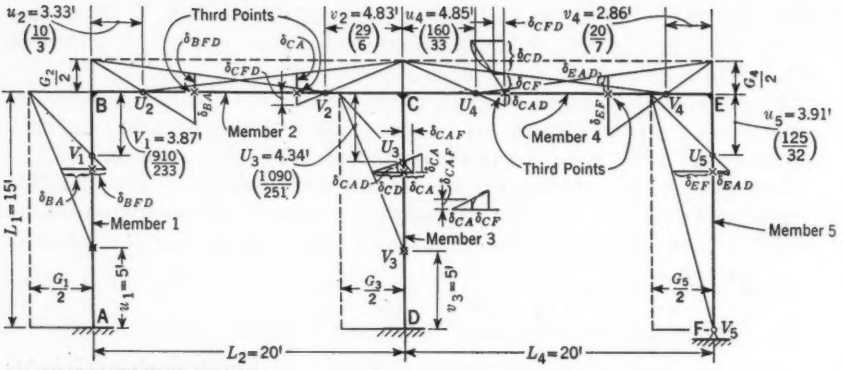
To take the lateral displacement of the column into account, the shears H_1, H_2, and H_3 are calculated from the moment diagram, and the sum, ΣH, of opposite sign, is applied at the temporary support, point E. The effect of this temporary reaction is obtained by allowing arbitrary translations $\Delta_B = \Delta_C = \Delta_E$ of the columns and calculating the shear in each column. The shear $H_{1\Delta}$ produced by translation Δ_B is calculated from $\Delta_B = H_{1\Delta}\,G'_1\,r^2_1$, in which G'_1 is the total elastic weight of column AB, restrained at point A and elastically supported at point B, and r_1 is the semiaxis of the ellipse of elasticity of this column. The elastic centroid and r_1 are found graphically as shown in Fig. 24(*c*). Simple relationships permit the computation of moments at points B and A, produced by $H_{1\Delta}$, as follows:

$$\bar{M}_{BA} = \frac{6\,\Delta_B\,v_1}{L_1\,G_1\,(L_1 - u_1 - v_1)} \qquad (49)$$

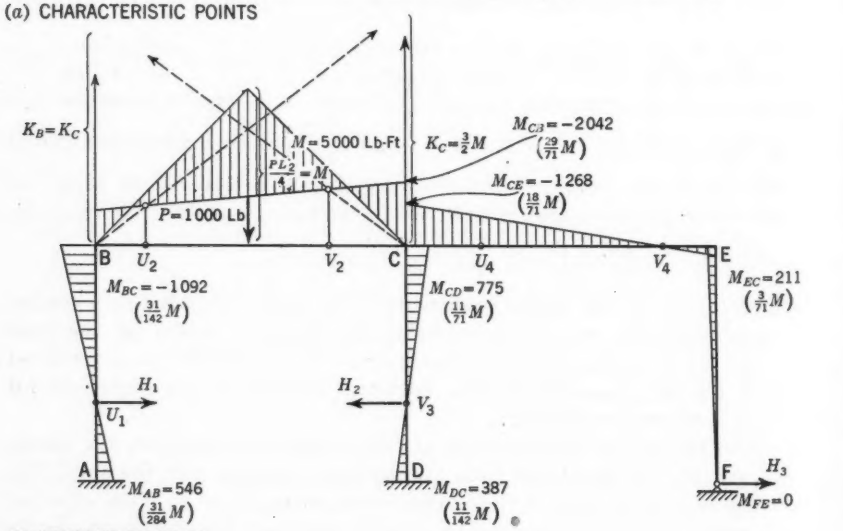
and

$$\bar{M}_{AB} = -\frac{6\,\Delta_B\,u_1}{L_1\,G_1\,(L_1 - u_1 - v_1)} = -M_{BA}\,\frac{u_1}{v_1} \qquad (50)$$

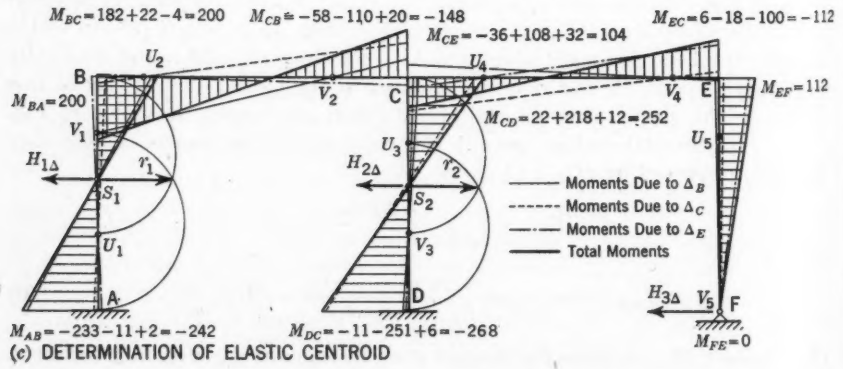
The moment $\bar{M}_{BA}$ is then distributed over all remaining structural members. Similarly, the moments due to $H_{2\Delta}$ and $H_{3\Delta}$, which are produced by $\Delta_C = \Delta_E$



(*a*) CHARACTERISTIC POINTS



(*b*) MOMENT DIAGRAM



(*c*) DETERMINATION OF ELASTIC CENTROID

FIG. 24.—GRAPHICAL SOLUTION OF EXAMPLE 2

$= \Delta_B$, are calculated. Finally, all moments produced by equal lateral displacement of the columns are summed up and the corresponding shear along member BCE is calculated.

Substituting the values in Eqs. 49 and 50: $\bar{M}_{BA} = \frac{182}{710}$ and $\bar{M}_{AB} = -\frac{233}{710}$. The moment $\bar{M}_{BA}$ is distributed to the other joints using the final carry-over factors which are determined by fixed points and angular displacements (at the joint C). The carry-over factor from point B to point C is $C_{BC} = \frac{v_2}{L_2 - v_2} = \frac{29}{91}$; the carry-over factor $\frac{M_{CD}}{M_{CB}} = \frac{11}{29}$; the carry-over factor $\frac{M_{CE}}{M_{CB}} = \frac{18}{29}$, etc. Similarly, the values $\bar{M}_{CD} = \frac{218}{710}$ and $\bar{M}_{DC} = -\frac{251}{710}$ (for column DC) and $\bar{M}_{EF} = \frac{112}{710}$ (for column FE) are obtained and carried over to the other joints (Fig. 24(*c*)). For the purpose of superposition, only the ratio between the temporary reaction at point E (Fig. 24(*b*)) and the total shear along member ECB (or the total thrust along this member) are considered. Therefore, the common denominator 710 can be eliminated (Fig. 24(*c*)). The temporary reaction at point E is $\frac{39}{284 \times 15} M$ (Fig. 24(*b*)) and the total shear produced by sidesway is $\frac{1,074}{15}$; hence, the ratio sought is $\frac{39 \times 5,000}{284 \times 1,074} = 0.63934$.

The additional moments produced by sidesway are calculated by multiplying the values in Fig. 24(*c*) by the ratio 0.63934.

Professor Wilson uses the moment distribution method (except in Example 1, in which the slope-deflection method is used) for the determination of the superimposing effects. The relative accuracy depends on the number of cycles and is usually completely sufficient for a rational design as shown by the values obtained by the author as compared with more accurate results of the writer's computations by the method of fixed (characteristic) points:

Example	Moments
2	$M_{AB} = 390$ (391.05); $M_{BA} = 963$ (963.68); $M_{CB} = 2,135$ (2,136.88); $M_{CD} = -933$ (−935.76); $M_{DC} = -557$ (−558.67); $M_{CE} = -1,202$ (1,201.11); and $M_{EC} = -137$ (−139.86).
4	$M_{AB} = -2,838$ (−2,831.37); $M_{BA} = -2,345$ (−2,325.14); $M_{CD} = -1,612$ (−1,612.93); $M_{DC} = -799$ (−808.81); $M_{DE} = -1,407$ (−1,403.79); $M_{ED} = -1,954$ (−1,955.97); $M_{DF} = 2,203$ (2,212.61); $M_{FG} = -1,109$ (−1,112.61); and $M_{GF} = -1,111$ (−1,112.14).
5	$M_{AB} = 8.71$ (8.79); $M_{BA} = 14.58$ (14.65); $M_{CB} = 2.70$ (2.82); $M_{CD} = 4.06$ (3.92); $M_{DC} = 6.30$ (6.31); $M_{ED} = -7.75$ (−7.75); $M_{FE} = 12.15$ (12.02); $M_{CF} = -6.76$ (−6.74); $M_{FC} = -2.66$ (−2.64); $M_{FA} = -9.49$ (−9.38); and $M_{AF} = -8.71$ (−8.79).

Example	Moments
6	$M_{AB} = 228$ (230.93); $M_{BA} = 428$ (429.18); $M_{CB} = 640$ (642.25); and $M_{DC} = -304$ (−304.79).
7	$M_{AB} = -472$ (−474.36); $M_{BA} = -433$ (−433.76); $M_{BE} = -134$ (−135.19); $M_{EB} = -308$ (−307.40); and $M_{BC} = 569$ (568.95).
8	$M_{AB} = 6{,}970$ (6,986.53); $M_{BA} = 3{,}740$ (3,746.31); $M_{CD} = 2{,}250$ (2,256.11); $M_{DC} = 550$ (539.99); and $M_{ED} = 3{,}800$ (3,831.97).

In the foregoing, the writer's values are in parentheses. The results are accurate, the differences being less than 4%. In Example 3, no differences were found.

Bibliography.—

(1) **"Analysis of Rigid Frames,"** by A. Amirikian, Government Printing Office, Washington, D. C., 1942. (*a*) Fig. 78, p. 109. (*b*) Fig. 104, p. 145.

(2) **"Analysis of Continuous Frames by Distributing Fixed-End Moments,"** by Hardy Cross, *Transactions*, Am. Soc. C. E., Vol. 96 (1932), pp. 1–156.

(20) **"Simplified Determination of Moments in Multi-Story Structures,"** by A. W. Fischer, *Concrete*, November, 1938, p. 5.

(21) *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 170.

(22) **"Relative Flexure Factors for Analyzing Continuous Structures,"** by Ralph W. Stewart, *ibid.*, Vol. 104 (1939), p. 521.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SIMULTANEOUS CONSOLIDATION OF CONTIGUOUS LAYERS OF UNLIKE COMPRESSIBLE SOILS

Discussion

BY EDWARD S. BARBER, AND JACOB FELD

EDWARD S. BARBER,[16] ESQ.[16a]—A timely analysis of the consolidation of contiguous layers of foundation soils having different consolidation characteristics is offered in this paper. The mathematically correct approach provides opportunity to evaluate the results furnished by the less laborious but approximate methods. An approximation that provides a reasonably simple solution for the consolidation of any number of contiguous layers and takes into account an influencing factor not considered in the author's method of average coefficients—namely, the relative position of the consolidating soil layers and the drainage faces—is outlined in this discussion. This approximation produces results that agree more closely with the exact theoretical analysis than do those by the method of average coefficients.

By considering the average gradient in the incompressible layer, H_d, shown in Fig. 1(*a*), required to carry away the water squeezed out of the consolidating layer, H, at any degree of consolidation, U, an approximate formula is obtained for the curves in Fig. 2, as follows:

$$U_R = \frac{U_\infty}{1 + \frac{U_\infty}{T\,R}} \qquad (37)$$

in which U_R = the value of U for a particular value of R, and U_∞ = the value of U when R is infinite, a theoretical condition assumed if it were possible to have the coefficient of permeability of the drainage layer, k_d, large enough to provide unrestricted flow.[17] The results furnished by Eq. 37 are good except for the combination of large values of T and small values of R.

NOTE.—This paper by Hamilton Gray, Jun. Am. Soc. C. E., was published in February, 1944, *Proceedings*.

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[16a] Received by the Secretary May 10, 1944.

[17] "The Theory of Soil Consolidation and Testing of Foundation Soils," by L. A. Palmer and E. S. Barber, *Public Roads*, March, 1937, p. 15.

The dimensionless coefficients, R, σ, μ, ν, and c used in Professor Gray's formulas may be simplified by expressing them in terms of the strain per unit increase in pressure, s, rather than a. The strain, s $\left(\text{which equals } \frac{a}{1+e}\right)$, can be obtained directly from the dial readings in a consolidation test. Actual thicknesses and permeabilities may be used instead of their reduced values and the void ratio becomes unnecessary.

Thus:

$$R = \frac{H/k}{H_d/k_d} \quad \ldots\ldots (38a)$$

$$\sigma = \sqrt{\frac{k_1 s_1}{k_2 s_2}} \quad \ldots\ldots (38b)$$

$$\mu \nu = \frac{H_2/k_2}{H_1/k_1} \frac{1}{\sigma} \quad \ldots\ldots (38c)$$

and

$$c_v = \frac{H^2 T}{t} = \frac{k}{\gamma s} \quad \ldots\ldots (38d)$$

Reference is now made to the comparison between the results furnished by the method of average coefficients and those by the exact analysis.

TABLE 4.—PROPERTIES OF LAYERS IN ILLUSTRATIVE EXAMPLE

Layer	s	H	k	H/k	sH
II	1	1	1	1	1
I	0.4	5	10	0.5	2
Total	....	6	....	1.5	3

It appears that the consolidation of several layers should be averaged on the basis of the settlements rather than the thicknesses, as shown in Eq. 35. In such case, the agreement between the exact and average methods would be even closer than that shown in Fig. 8. The agreement of the two methods discussed by the author may not always be as close as that indicated in Fig. 8. To illustrate, assume two contiguous consolidating layers with one drainage face and properties as given in Table 4.

Using the method of average soil coefficients outlined by Eq. 36 and taking $\gamma = 1$: $k = \frac{\sum H}{\sum H/k} = 4$; $s = \frac{\sum s H}{\sum H} = 0.5$; $c_v = \frac{k}{\gamma s} = 8$; and $t = \frac{H^2}{c_v} T = 4.T.5$

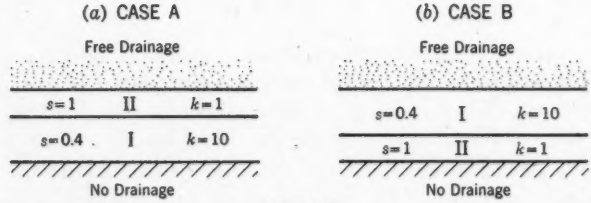


FIG. 10

Using Professor Gray's exact method, there are two cases (see Fig. 10): Case A, impervious face next to layer I, and case B, impervious face next to

layer II. The constants are:

Constant	Case A	Case B
c_v	25	1
t/T	1	1
σ	2	0.5
$\mu\ \nu$	1	1

The time-consolidation curves resulting from these two methods of calculation are shown in Fig. 11. Also, in Fig. 11 are shown the curves obtained by means of the approximation introduced by the writer (which modifies the

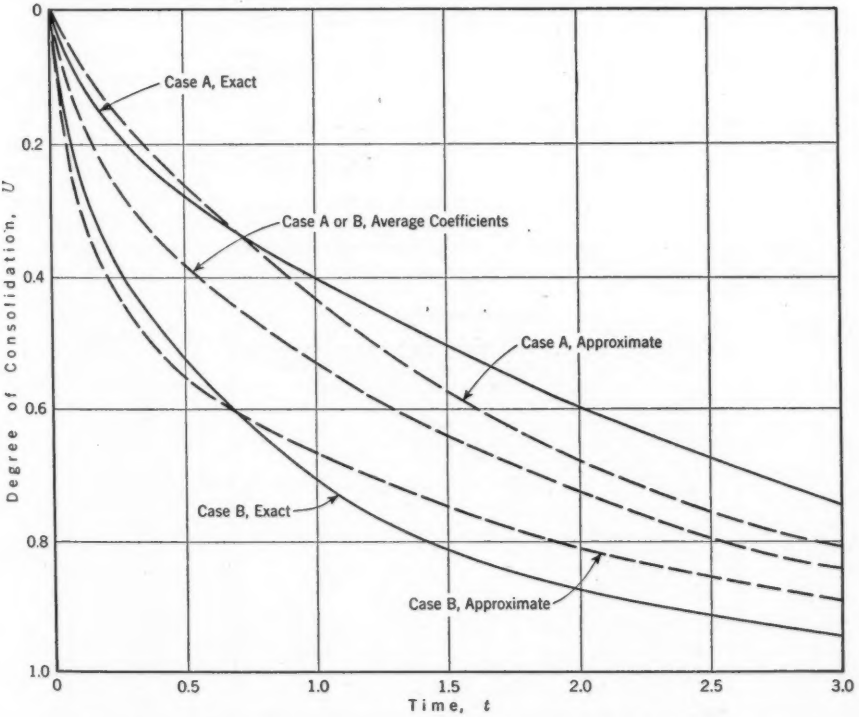


FIG. 11.—COMPARISON OF COMPUTATIONS OF TIME-CONSOLIDATION IN COMBINED LAYERS

method of average coefficients to take into account the effect of the relative position of the boundaries and thus agrees more closely with the exact method).

The effect of this approximation is illustrated by the example for three contiguous layers outlined in Table 5. The time is computed as in the method of average coefficients. Thus, using the relationship,

$$s\,H = \frac{\delta}{p} \qquad (39)$$

the elapsed time t is expressed by

$$t = T \sum \frac{\delta}{p} \sum \frac{H}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

which becomes 10 T for $p = 1$. However, the actual layers are replaced by their "equivalent thicknesses" H/k (Col. 4, Table 5) and, instead of using the

TABLE 5.—EXAMPLE OF THREE CONTIGUOUS LAYERS

Given				Computation				Inches of Settlement in Layer (Eq. 41) for the Following Degrees of Load Transfer U			
Layer No.	Final settlement δ (in.)	Thickness of layer H (ft)	Coefficient of permeability k	Equivalent thickness H/k	Equivalent part of total thickness H_p	Limits of Layer (Total H_p)					
						From:	To:	0.20	0.40	0.60	0.80
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	3	6	1	6	0.30	0	0.30	0.96	1.67	2.10	2.50
2	1	20	2	10	0.50	0.30	0.80	0.03	0.19	0.46	0.74
3	2	12	3	4	0.20	0.80	1.00	0.88	1.38	1.60	1.80
....	6	38	....	20	1.00[a]	....	....	1.87	3.24	4.16	5.04

[a] This is for drainage above and below. For an impervious lower boundary make this value 0.50.

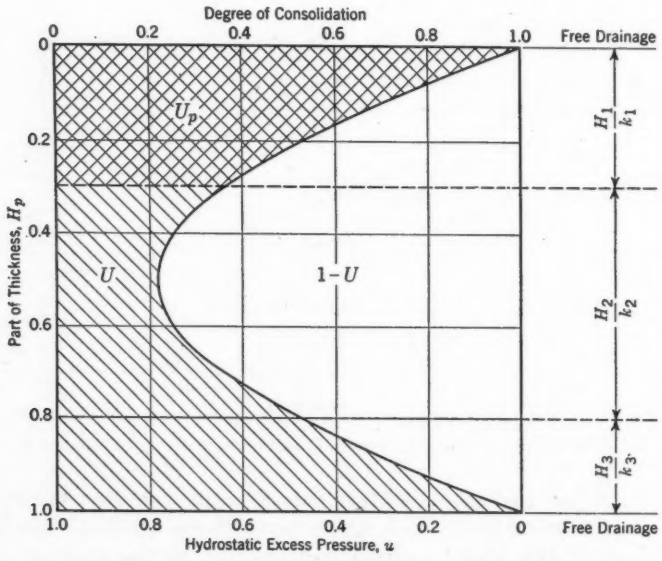


FIG. 12.—APPROXIMATE ANALYSIS OF TIME-CONSOLIDATION IN COMBINED LAYERS

same percentage load transfer in each layer, the part of the load transfer in each part of the total thickness[18] is used as shown in Fig. 12 and Table 6.

[18] "Notes on Analytic Soil Mechanics," by J. R. Bruggeman. C. N. Zangar, and J. H. A. Brahtz, *Technical Memorandum No. 592*, Bureau of Reclamation, U. S. Dept. of the Interior, 1939, Fig. 34.

In Table 5, Cols. 8 to 11, the settlement in a layer is computed by

$$\delta_p = \delta \frac{\Delta U_p}{H_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

in which values of ΔU_p are taken from Table 5 as the difference between the

TABLE 6.—CALCULATION OF TIME-CONSOLIDATION IN COMBINED LAYERS

(Degree of Load Transfer in Layer, U_p)

Part of total thickness H_p	DEGREE OF LOAD TRANSFER IN TOTAL THICKNESS, U:								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.028	0.037	0.044	0.046	0.047	0.048	0.048	0.049	0.049
0.10	0.042	0.062	0.077	0.084	0.087	0.091	0.093	0.095	0.098
0.15	0.046	0.081	0.103	0.113	0.123	0.129	0.134	0.140	0.145
0.20	0.048	0.088	0.121	0.138	0.149	0.160	0.171	0.180	0.191
0.25	0.049	0.095	0.133	0.156	0.174	0.189	0.205	0.219	0.236
0.30	0.049	0.096	0.139	0.167	0.190	0.210	0.231	0.250	0.274
0.35	0.050	0.098	0.144	0.180	0.210	0.236	0.264	0.292	0.322
0.40	0.050	0.100	0.147	0.188	0.224	0.258	0.294	0.328	0.365
0.45	0.050	0.100	0.149	0.194	0.236	0.278	0.320	0.360	0.405
0.50	0.050	0.100	0.150	0.200	0.250	0.300	0.350	0.400	0.450
0.55	0.050	0.100	0.151	0.206	0.264	0.322	0.380	0.440	0.495
0.60	0.050	0.100	0.153	0.212	0.276	0.342	0.406	0.472	0.535
0.65	0.050	0.102	0.156	0.220	0.290	0.364	0.436	0.508	0.578
0.70	0.051	0.104	0.161	0.233	0.310	0.390	0.469	0.550	0.626
0.75	0.051	0.105	0.167	0.244	0.326	0.411	0.495	0.581	0.664
0.80	0.052	0.112	0.179	0.262	0.351	0.440	0.529	0.620	0.709
0.85	0.054	0.119	0.197	0.287	0.377	0.471	0.566	0.660	0.755
0.90	0.058	0.138	0.223	0.316	0.413	0.509	0.607	0.705	0.802
0.95	0.072	0.163	0.256	0.354	0.453	0.552	0.652	0.751	0.851
1.00	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900

TABLE 7.—VALUES OF TIME FACTOR T; EFFECT OF BOUNDARIES ON TIME-CONSOLIDATION

f[a]	DEGREES OF CONSOLIDATION U								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.049	0.100	0.154	0.217	0.29	0.38	0.50	0.66	0.95
0.2	0.027	0.073	0.126	0.186	0.26	0.35	0.46	0.63	0.92
0.4	0.016	0.056	0.106	0.164	0.24	0.33	0.44	0.60	0.90
0.6	0.012	0.042	0.092	0.148	0.22	0.31	0.42	0.58	0.88
0.8	0.010	0.036	0.079	0.134	0.20	0.29	0.41	0.57	0.86
1.0	0.008	0.031	0.071	0.126	0.20	0.29	0.40	0.56	0.85
1.5	0.006	0.024	0.058	0.107	0.17	0.26	0.38	0.54	0.83
2.0	0.005	0.019	0.050	0.095	0.16	0.24	0.36	0.52	0.81
3.0	0.004	0.016	0.041	0.082	0.14	0.22	0.34	0.50	0.79
5.0	0.003	0.013	0.034	0.069	0.12	0.20	0.32	0.48	0.77
10.0	0.003	0.011	0.028	0.060	0.11	0.18	0.30	0.46	0.75
∞	0.002	0.009	0.024	0.048	0.09	0.16	0.28	0.44	0.73
....[b]	0.002	0.008	0.018	0.031	0.05	0.07	0.10	0.14	0.21

[a] Pressure at drainage face divided by pressure at impervious face = f. [b] Values of time factor for two drainage faces.

U_p-values at the limits of the layer. Thus, for layer 2 (see Col. 10, Table 5), when $U = 0.60$ the value of ΔU_p (for $H_p = 0.30$ to 0.80) is equal to 0.440 − 0.210 = 0.230 and δ_p (see Eq. 41) is equal to $1 \times \frac{0.23}{0.50} = 0.46$.

The writer has assumed that the increase in pressure applied to the consolidating layers is uniform throughout their total thickness. To include the effect of a trapezoidal distribution of the initial excess pressure, values of T may be taken from Table 7, which is based on the following interpolation formula:[19]

$$U = U_1 + \frac{f-1}{f+1}(U_1 - U_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

in which T is constant and the subscripts are values of f.

JACOB FELD,[20] M. AM. SOC. C. E.[20a]—By clearly indicating how the limitation of actual field conditions affects the theoretical solutions, the author has provided a valuable method for determining the range of values, not only in magnitude but also in rate of occurrence. This procedure will not mislead the engineer who looks to such tools for guidance in the determination of expected consolidation in soil fills and under foundations. All theoretical solutions are based on assumed end and surface conditions. Unless such ideal conditions occur, the formulas can only approximate true values and probably only give a range within which results may be expected. Only when a prophecy of consolidation, both as to extent and rate, is made on the basis of theoretical conditions alone, is no correlation found between actual occurrence and expected values, and one loses faith in all quantitative soil studies.

Sudden clear-cut changes in soil layers are not often encountered in nature —even in rock formations. There is usually a conglomerate, mixed skin separating unlike soil layers. Such a condition is difficult to evaluate in theoretical solutions, and thus subjects of this nature fall in the class of the practical art rather than in that of the theoretical science.

The same problem occurs in building up road and airport surfaces on subbase materials. When a layer of pervious material is placed directly on a clay subgrade, apparently the ideal condition of contact in soil layers occurs. Actually, experience indicates that the two materials mix and form an intervening skin, which is similar to neither of the parent layers. An accelerated field demonstration of this phenomenon (probably not new) occurred during the construction of some twenty-five miles of pavements on extremely fine and impervious Seneca County (New York) clays in 1942. An examination of all existing roads in that area, and experience on other projects built in the previous year, showed that the fine clay worked up through the run-of-bank gravel sub-base, through penetration macadam, and even through bituminous mix surfacing. In one year, under rather light traffic, the clay had come through a total thickness of 8 in. of gravel sub-base, 4 in. of crushed dolomite base, and a penetration surfacing.

However, inspections, after almost two years of use, on the roads built under the writer's supervision showed no vertical intrusion of the clay into the road construction. In these cases, a fine bank sand containing leachings from

[19] "The Theory of Soil Consolidation and Testing of Foundation Soils," by L. A. Palmer and E. S. Barber, *Public Roads*, March, 1937, p. 18, Eqs. 53, 54, and 55.

[20] Cons. Engr., New York, N. Y.

[20a] Received by the Secretary May 18, 1944.

limestone deposits had been placed about 2 in. deep on the excavated roadbed. Over the sand 6 in. to 8 in. of run-of-bank gravel, 4 in. of crushed stone base, and 2½ in. of tar mix surfacing were then placed. Similarly, some test roads built directly on top of a 2-in. or 3-in. weathered clay "topsoil" containing some grass roots were also satisfactory. Apparently the introduction of a fine sand screen, rolled into the clay and forming a conglomerate skin, changes all the consolidation characteristics normally encountered in the clay.

A consolidation forecast of filled materials must also take into account the possible altering of the exposed surface with the formation of a crust which will entirely change the expected free drainage surface conditions. In the construction of a large airport, grading operations included the removal of glacial till hills and the filling in of valleys to depths of almost 60 ft. The expected shrinkage and consolidation did not occur. When the paving was placed, about a year later, it was found necessary to remove about 1 ft of the fills to bring the surface to subgrade elevations. This top crust was very hard and of high density, but was floating on an almost liquid material which had not consolidated. The result could be prophesied by the author's methods if it had been assumed that there was practically no surface evaporation. However, if that assumption is not made, an erroneous result is found from the use of any consolidation formula.

In placing fills over completed subway and similar structures in densely populated areas, the most important requirement is rapid consolidation. Otherwise, there is not sufficient time to replace the subsurface utility structures in such a manner that failures will not occur after the streets are repaved. Too little has been done in experimenting with artificially treated backfill materials, when ideal porous filling materials are not available. The necessity for such treatment can be determined from the methods developed by the author, and by proper assumptions the relative depths of treated and untreated fills can also be obtained. In one personal experience, the use of river silt taken from a subaqueous tunnel operation was found unsatisfactory because of the slow drainage characteristics. However, when the tunnel operation ran into some difficulties and an appreciable amount of hay was used for "mudding up" the heading of the tunnel, the silt material became excellent backfill material and consolidated very rapidly. Once it was consolidated there was little danger of future volume changes because of the restrictions of subsurface structures, foundations, and impervious street paving. Of course, this idea is ancient in origin, dating back to biblical, or earlier, days when bricks were manufactured by the addition of straw to clay.

The author is to be congratulated for the presentation of workable formulas which can be checked in the field and which will give consistent and true results if, and only if, consistent and true field conditions are evaluated (as he so carefully explains) before using these formulas.

Correction for *Transactions:* February, 1944, *Proceedings*, page 149, in the ordinate captions of Figs. 6 and 9, change "*Q*" to "*U*."

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ERECTION OF HYDRAULIC TURBINES AND GENERATORS

Discussion

BY PAUL E. GISIGER, AND FRANK I. MORGAN

PAUL E. GISIGER,[5] M. AM. SOC. C. E.[5a]—A useful summary is presented of the main points to be observed in the erection of Kaplan and Francis wheel hydroelectric generating units. The writer would like to add one recommendation to this competently presented paper—namely, that those who are in actual charge of erection make and keep a permanent record of all elevations, clearances, tolerances, etc., established at the time of erection. This record should be checked and if necessary supplemented immediately before the unit goes into regular operation.

The writer has several times regretted the lack of such a record. It is fairly obvious that the levels and tolerances established during erection of a hydroelectric unit are not permanent. Even the best of concrete is subjected to slight changes of volume and shape due to the factors of shrinkage and creep, apart from temperature variations and elastic deformations. The dislocations caused by these factors are generally of equal, if not higher, order of magnitude than the tolerances within which a unit can be erected. The resulting changes in alinement are in most cases inconsequential and may be unnoticed for years, but occasionally they are large enough to require correction and then the question arises as to how much change has taken place since the time when the unit was erected. The horizontal alinement of a generator stator may have changed, the runner blades of a Kaplan turbine may have left a trace on some sector of the circumference of the discharge ring, a bearing may show signs of pressure concentrated on one side, or a shaft may be found to be slightly out of plumb. In each case several causes may be responsible for the change. Without an accurate record for reference it is difficult, if not impossible, to judge the nature and causes of the trouble and to apply suitable remedies.

NOTE.—This paper by Andrew M. Komora, M. Am. Soc. C. E., was published in March, 1944, *Proceedings*.

[5] Structural Engr., Pennsylvania Water and Power Co. and Safe Harbor Water Power Corp., Baltimore, Md.

[5a] Received by the Secretary May 4, 1944.

There are normally at least three parties concerned with the erection of a hydroelectric unit—the purchaser, the turbine manufacturer, and the generator manufacturer. If something does get out of plumb or alinement, it is natural for the equipment manufacturers to question the structure on which this equipment is supported. All the interested parties then need a complete record of the condition of the unit at the beginning of its useful life.

From this viewpoint it cannot be recommended too strongly that a complete record be kept of each unit, giving all key measurements and elevations such as, for instance, diameters of the discharge ring in at least four directions, elevations of marked points of the speed ring, elevations of the points of support of the main bearing brackets, and elevations of the stator base plates. In addition to these points on parts of the generating unit itself, it will be helpful to maintain a record of observation points that permit following any slight change in the structural supports. Such points may be located, for instance, on the two main diameters at the lowest accessible points of the pit liner and at corresponding locations higher up on the concrete foundation of the generator stators. The cost of installing such observation points or bench marks at the time of erection is insignificant when compared to the value of the information they may reveal during the life of the unit.

FRANK I. MORGAN,[6] ESQ.[6a]—A longfelt need has been met by the publication of this interesting paper. The writer wishes to add his comments in the hope that Mr. Komora will clarify a few points. For example, under "Stage 1. Placing and Erecting Turbine Embedded Parts," the author states: "Whether a level or transit is used [in setting equipment] depends on the engineer's preference." Is it not true that there is no choice since the transit is not a sufficiently accurate instrument for this purpose? The use of a high-grade level is quite essential to insure accurate results.

The author mentions difficulties experienced with the I-beam type of bearing bracket because of improper shimming, etc., of the side arms. This has been a serious problem and much trouble has been experienced with guide bearings because distortion of the bearing retainer resulted from increased deflection under load. This defect has been corrected in later designs by hinging the side arms. It would be interesting to learn how the bearing clearance is determined, with the machine under load as recommended by the author.

Mr. Komora mentions the use of four plumb lines when plumbing the shaft. Why are four lines necessary? It should be possible to obtain all the data with two plumb lines 90° apart—reducing the amount of work considerably.

The author states that it is customary to make the coupling bolts 1 to 2 mils oversize. What are the particular benefits of this procedure? This practice requires considerable extra time and labor during subsequent dismantling work. Furthermore, such tight fitting bolts tend to distort the faces of the flanges and to make difficult the accurate determination of the stress, in the bolts, during the process of tightening. A light tapping fit for coupling bolts is

[6] Maintenance Engr., Pennsylvania Water and Power Co. and Safe Harbor Water Power Corp., Baltimore, Md.

[6a] Received by the Secretary May 4, 1944.

considered adequate. If the bolts are properly stressed when tightened, the friction between the coupling faces should be sufficient to transmit the torque. In one particular case the bolt holes in one flange of a shaft coupling were inadvertently reamed oversize. The unit, a large one, was assembled with the coupling in this condition, and an inspection quite some time after the unit had been in service, under load, showed that no movement had taken place in the coupling.

Mr. Komora is quite emphatic in his condemnation of the practice of shipping the turbine shaft to the generator manufacturer, there to be assembled with the generator shaft and the entire assembly checked for truth in a lathe. Such procedure is advisable until the erection of large units, on which this course has not been followed, has proved satisfactory. In at least one case, such a check revealed considerable inaccuracy which could be corrected only by remachining. Had the shafts reached the field, serious delay would have resulted. Shop assembly also makes it unnecessary to ream coupling bolt holes in the field.

In Table 1 the actual throw of a particular shaft is given as 0.0055 in. The author states (see heading, "Stage 5. Erection of Generator-Mechanical Features"), "The center of this circle [the shaft throw] is 2 mils south and $1\frac{1}{4}$ mils west from the theoretical center line. * * * The shaft was replumbed so that the final position was 1 mil west and zero mils north and south." Five mils is extremely small for the throw of a shaft, and since the center of the throw is but 2 mils away from the true center the position of the shaft is almost ideal—making any changes in its location somewhat inconsistent with the author's statement that much unnecessary time and effort has been spent in alining units. In this particular case the clearance of the guide bearing in which this shaft operates is probably 12 or 16 mils; therefore, the throw of the shaft plus the error in the position of the center would still be small enough to be within the bearing clearance. Units having shaft throws of 50 mils are operating successfully although this is, by no means, a desirable condition.

Has Mr. Komora had experience in plumbing the shafts and checking the throw of units equipped with spring type thrust bearings? With this type of thrust bearing the shaft may be made absolutely plumb and centered in the turbine guide bearing. As soon as the rotor is turned, however, the shaft will no longer be plumb because of unequal yielding of the springs in the thrust bearing. Thus, the final setting must be a compromise.

The author states that a period of several weeks has been spent in alining units having shrunk hubs. If the surface of the hub (more properly, the thrust block) is not normal to the shaft axis, how can the unit be alined correctly since no amount of adjustment of the thrust bearing will eliminate this error?

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

RIGIDITY AND AERODYNAMIC STABILITY OF SUSPENSION BRIDGES

Discussion

BY F. B. FARQUHARSON

F. B. FARQUHARSON,[45] M. AM. SOC. C. E.[45a]—An extremely able mathematical treatment of the problem of aerodynamic stability of suspension bridges is contained in this paper. It appears, however, that some of the data available to the author relating to the behavior of the original bridge were in error and that the author was not informed on some of the uniform conclusions from laboratory tests conducted since the failure of the Tacoma Narrows Bridge. It is unfortunate that the pressure of wartime has made it impossible to release the results of certain very extensive investigations which, although not complete, clarify some important phases of the problem. The following discussion refers to corresponding parts of the paper.

Parts 2 and 3.—One of the most important errors in the information available to the public regarding the behavior of the Tacoma Narrows Bridge lies in the published log of observed modes of motion and associated wind velocities.[3] A most serious error arose in the velocities obtained with an anemometer for which no calibration curve existed. Subsequent calibration of this instrument revealed errors of about 50% in the lower velocity range.

No uniform procedure was followed in obtaining these readings of velocity. Many of them were taken on the top of the toll house, which was 1,400 ft distant from the nearest tower and where the influence of the terrain was such, in southerly winds, as to alter, completely, the velocity as well as the direction of the wind. Many other readings were taken just beyond the first tower on the main span and again offered no reliable indication regarding the average wind on the bridge as a whole. Furthermore, none of the direction readings was obtained with any accuracy. Each reading was a pure estimate, and no

NOTE.—This paper by D. B. Steinman was published in November, 1943, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1944, by Edward Adams Richardson; and March, 1944, by R. K. Bernhard; and May, 1944, by J. M. Robertson, and C. H. Gronquist.

[45] Prof., Civ. Eng., Univ. of Washington, Seattle, Wash.

[45a] Received by the Secretary May 11, 1944.

[3] "The Failure of the Tacoma Narrows Bridge," a report to the Hon. John M. Carmody, Administrator of the Federal Works Agency, Washington, D. C., March 28, 1941, by a board of engineers consisting of O. H. Ammann, Theodor von Kármán, and Glenn B. Woodruff, Appendix VI, p. 19.

close correlation was maintained between observations of velocity and direction and readings of amplitude. In short, these recordings should be treated as only very sketchy records of the behavior of the wind whose true characteristics must remain forever in doubt.

The net result of the aforementioned discrepancies is that all of the velocity and direction readings in the log are subject to a correction. In general, the velocity error revealed by the anemometer calibration tapers from approximately 50% in the lower range to zero at about 30 miles per hr. It is impossible to estimate the magnitude of the errors due to location of the reading instrument, but it can be stated confidently that many of the readings were very low for that reason.

During the months following the failure of the Tacoma Bridge extensive research was undertaken in the Structural Research Laboratory at the University of Washington in Seattle under the joint sponsorship of the Washington Toll Bridge Authority and the Public Roads Administration. This program involved the building of a wind-tunnel laboratory capable of testing a 1 : 50 scale model of a suspension bridge of the proportions of the one lost at Tacoma Narrows. To date (June, 1944) two complete models have been built, the first a duplicate of the original bridge and the second in connection with the proposed new design. In conjunction with these full model studies, three section models have been constructed as a part of the investigation of the proposed new design for a bridge at Tacoma Narrows.

Paralleling this work at the University of Washington, a program was sponsored by the Washington Toll Bridge Authority at the California Institute of Technology under the direction of Theodor von Kármán, M. Am. Soc. C. E., using section models.

The difference between section and full models should be clarified. The full model is a complete duplication of the prototype, in every significant detail, and includes towers, cables, suspenders, and suspended structures constructed to appropriate model scale. The section model is composed of part of the suspended structure mounted on springs in such a manner that the various degrees of freedom existing on the prototype can be duplicated.

Several significant conditions have been revealed by the tests embraced in the two programs, and, although full correlation between section and full model has not yet been evaluated, on certain basic facts there is full agreement. With respect to the original Tacoma Narrows Bridge the most pertinent of these facts may be outlined as follows:

(*a*) A linear relationship exists between wind velocity V_c, corresponding to the velocity at which motion starts in a given mode, and the frequency of the several vertical modes.

(*b*) Each vertical mode appears twice but at the same frequency. The $V_c/(N\,b)$-value for the second appearance is twice that of the first. From this it follows that for a given configuration the critical velocity for the second appearance of any mode is twice that associated with the first appearance.

(*c*) For any given shape of the suspended structure and angle of attack of the wind, the value of $V_c/(N\,b)$ is constant for all vertical modes in their first

appearance and constant at twice that value for all vertical modes in their second appearance.

(*d*) The first appearance of each vertical mode is very weak and difficult to isolate on the full model. These first appearances are readily observed on the section model.

(*e*) Each configuration and angle of attack of the wind has its own constant value of $V_c/(N\,b)$.

(*f*) The torsional modes also appear twice and at the same frequency. The first appearance is weak and damps out with increasing velocity, but the second appearance which occurs almost immediately following the damping of the first becomes catastrophic with increasing wind velocity. In general, the critical velocity for the first appearance of a torsional mode is half that of the second appearance. Some of the tests involving alterations in configuration suggest that this relationship may not always hold.

Oscillation Coefficients.—In aeronautical literature associated with the wind flutter problem the dimensionless ratio $V/(N\,b)$ is sometimes called the "reduced velocity." This ratio is also referred to as a "reduced wave length," the wave occurring in the wake. Some investigators have found it convenient to refer to the reciprocal of this ratio as a "reduced frequency."

Since the ratio is dimensionless and may be viewed as the ratio of two velocities, two distances, or two times, depending on the point of view of the individual or the requirements of the problem, it seems poor practice to limit the meaning of the label by making it too specific. It is proposed that in the developing field of suspension bridge aerodynamics the $V/(N\,b)$-ratio be called the "oscillation coefficient." A parallel practice has already been adopted by Theodore Theodorsen when he refers to the critical "flutter coefficient."[46]

A part of the information in Items (*a*) to (*f*) is in Fig. 15 which is obtained from the results of tests under several radically different conditions. In view of this evidence, Table 3 should not be accepted as valid since it involves a variation of $V_c/(N\,b)$ over a range of 2.00 to 0.25. Laboratory results indicate that this value is constant in the vicinity of 2.20 for the second appearance of the vertical modes and in the vicinity of 1.09 for the first appearance.

The linearity of critical velocity with frequency is completely confirmed by section model tests at the California Institute of Technology (unpublished report), which of course confirm as well the constant value of $V_c/(N\,b)$ for a given configuration.

In Example 5, the author's check for high velocity is good, but the minimum velocity check is in error by a factor of at least five. The writer has not found the time for a critical examination into the possibility of making this mathematical approach fit the known facts more exactly. It is to be hoped that the author may be able to do so.

If the tests reported in Figs. 15 and 16 had been extended to higher wind velocities and if means had been provided to prevent the development of the torsional mode, successive higher vertical modes would have developed. (In

[46] "The Mechanism of Flutter, a Theoretical and Experimental Investigation of the Flutter Problem," by Theodore Theodorsen and I. E. Garrick, *Report No. 685*, National Advisory Committee for Aeronautics, 1940.

Fig. 16, N = frequency in cycles per second.) Following this procedure, motions involving eleven nodes have been noted with no indication that higher modes were not possible. The highest mode involved frequencies in excess of 360 cycles per min (since all field frequencies must be multiplied by the square root of the scale factor this would correspond to approximately 51 cycles per min in the field). These high frequencies were especially severe on the model. Therefore, they were never maintained for any long period of time and thus were never open to careful analysis.

The writer feels that there is no evidence to substantiate statement (*a*) (see heading in Part 2: "Conclusions Relative to Vertical Aerodynamic Oscillations"). There is every indication that vertical modes could have developed well above the eight-noded mode observed in the field at 42 miles per hr. If N is assumed at 0.85 cycles per sec for eleven nodes on the prototype and a value of $V_c/(N\,b) = 2.20$ is assumed from Fig. 15, then $V_c = 2.2 \times 0.85 \times 40 = 75$ ft per sec or 51 miles per hr.

In view of the accumulated evidence, the writer cannot agree with statement (*b*) either, since the evidence from both section and full model tests indicates that torsional tendencies are present on this structure from the time the velocity reaches about 6 miles per hr.

It should also be emphasized that torsional motions can develop on a section having a positive slope on the torque graph. Fig. 17 shows torque graphs for a large model of the original bridge which was tested at both the California Institute of Technology and the University of Washington. These curves show that a full circular fairing applied to the leading and trailing girders reversed the slope of the torque graph in both tests; yet when the full model was tested in this form the amplitude response curves were similar to those of Fig. 16. Recent section model tests at the University of Washington on sections involving trusses have shown a decided torsional motion under certain small angles of attack although the torque graph showed a definite positive slope.

The chief differences between the faired and the unfaired sections seem to be that on the faired section the $V_c/(N\,b)$-value for the vertical modes is considerably lower, and the maximum amplitudes developed in the various modes somewhat lower, than on the unfaired bridge. The degree of torsional instability was somewhat altered, and there is reason to believe that the second appearance of torsion may have damped out again after reaching an angle of about 10° and a wind velocity of approximately 20 miles per hr. These tests were run just before the model was dismantled. They were rather hurried and were never repeated for confirmation. Therefore, they should be taken as indications only. The test was not extended far enough to determine whether torsion appeared again at higher wind velocities.

No attempt has been made recently to explain the vertical motions observed on the basis of the static wind tunnel tests—that is, on the basis of negative aerodynamic damping. Experiment has demonstrated that only very slight differences showed in vertical modes between a section that had a definitely negative slope on the lift line (say, a d/b-ratio of 0.35) and the section of the original bridge (a d/b-ratio of 0.2) that showed a positive slope. Thus other

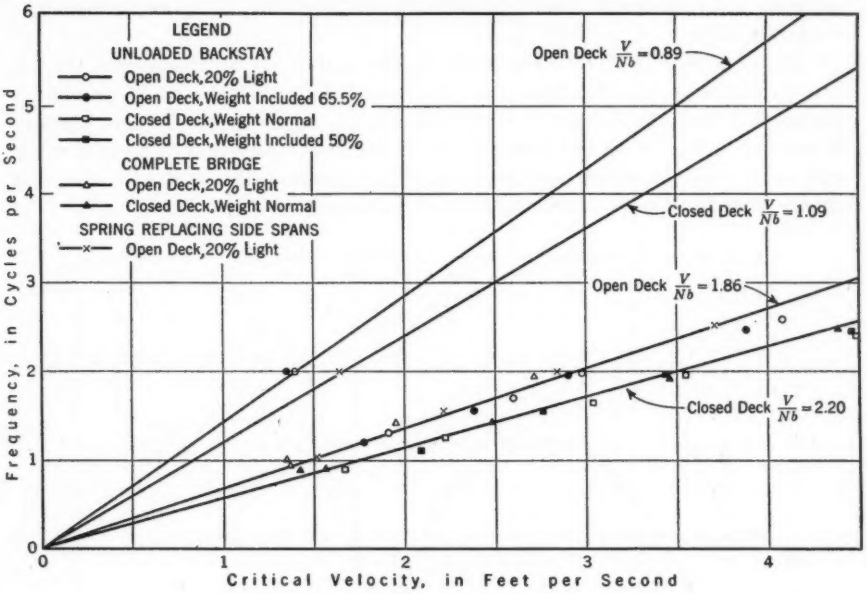


FIG. 15.—VERTICAL OSCILLATIONS

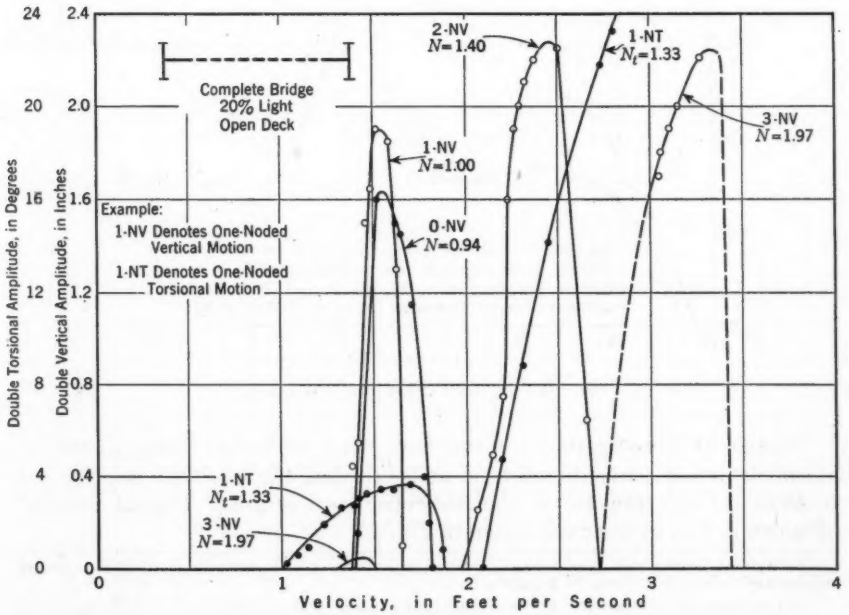


FIG. 16

possible sources have been investigated anticipating a more satisfactory explanation.

Early studies had led to the setting aside of the "vortex discharge" theory; but subsequent investigations have indicated that this theory is tenable in connection with the excitation of vertical modes on the basis of "subharmonic resonance"—that is, it has been demonstrated that excitation of a given mode can occur under starting conditions involving a vortex discharge at a rate that shows some multiple of the frequency of the resulting oscillation.

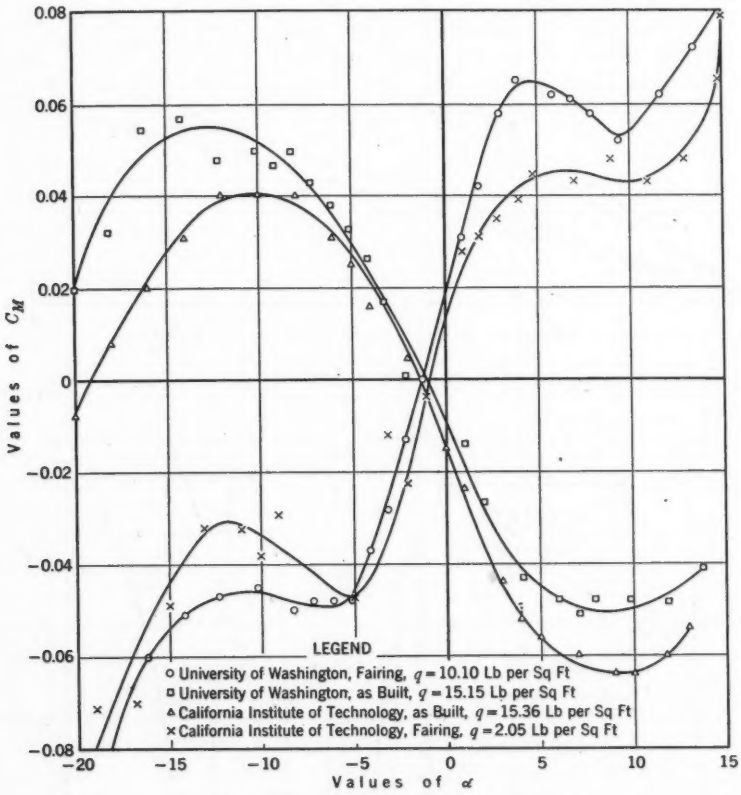


FIG. 17.—MOMENT COEFFICIENT, MODEL D

Numerous investigations of the flow were conducted using a hot-wire anemometer[47] of conventional form and recording the resultant velocity fluctuations through the use of a piezoelectric oscillograph. Typical records[47a] obtained in this manner are shown in Fig. 18.

[47] "Applied Hydro- and Aeromechanics," based on lectures of L. Prandtl, by O. Tietjens, McGraw-Hill Book Co., Inc., New York, N. Y., 1934.

[47a] Correction for *Transactions:* In the six subcaptions of Fig. 18 the expression "$V = \frac{1.52}{S}$" etc., should read "V = 1.52 ft per sec," etc.

As a result of these the hot-wire tests made simultaneously at the University of Washington and the California Institute of Technology (the latter used a different means of recording), it is now felt that a reasonable explanation of the phenomenon is available.

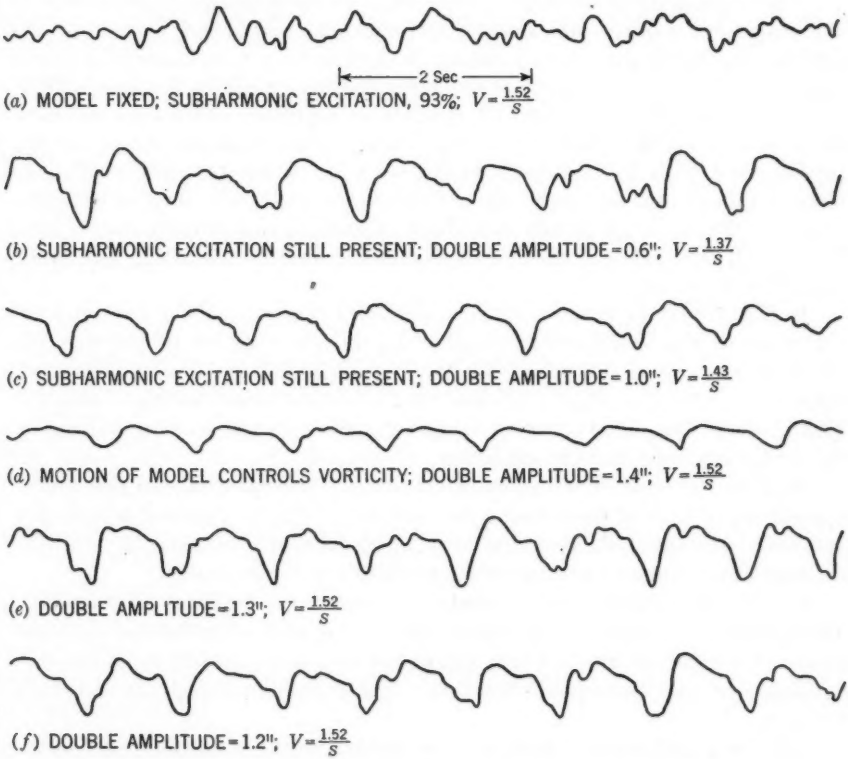

2 Sec
(a) MODEL FIXED; SUBHARMONIC EXCITATION, 93%; $V=\frac{1.52}{S}$
(b) SUBHARMONIC EXCITATION STILL PRESENT; DOUBLE AMPLITUDE=0.6"; $V=\frac{1.37}{S}$
(c) SUBHARMONIC EXCITATION STILL PRESENT; DOUBLE AMPLITUDE=1.0"; $V=\frac{1.43}{S}$
(d) MOTION OF MODEL CONTROLS VORTICITY; DOUBLE AMPLITUDE=1.4"; $V=\frac{1.52}{S}$
(e) DOUBLE AMPLITUDE=1.3"; $V=\frac{1.52}{S}$
(f) DOUBLE AMPLITUDE=1.2"; $V=\frac{1.52}{S}$

FIG. 18.—TYPICAL VELOCITY FLUCTUATIONS, SINGLE-NODED VERTICAL MODE, WITH $N=1.0$ CYCLES PER SECOND

As the wind strikes the section that is at rest, the rate of vortex discharge from the leading girder will be indicated at a multiple of some natural frequency of the structure, and the structure will begin to oscillate in an appropriate mode. If the velocity is maintained at a value close to the critical velocity for that mode, the amplitude will remain small and excitation will continue on a subharmonic basis. As the velocity is increased, however, the section begins to take charge of the flow, and subharmonic excitation yields slowly to harmonic excitation.[48] At some velocity, generally well below that associated with the maximum steady state, the excitation becomes harmonic and the vorticity is completely in phase with the motion. In other words, the section has taken complete charge of the flow. As velocity is increased still further, the section

[48] "Some Experimental Results on Wing Flutter," by William Bollay and C. D. Brown, *Journal of the Aeronautical Sciences*, June, 1941, pp. 313–318.

begins to lose its domination over the flow, the amplitude falls off, and the section approaches rest in so far as that particular mode is concerned. In general, there is no exact rest point in the amplitude response picture for any configuration since there is a certain overlapping of the modes.

The domination of the flow by the section seems somewhat tenuous even at the maximum steady state since it is frequently noted that amplitude will decrease markedly without any change in velocity only to increase as constant velocity is maintained. In the case of the lower vertical modes especially, it has often been difficult to build the maximum amplitude under what appeared to be exactly similar conditions. For example, the response curves for the fundamental (0-N V) and single-noded (1-N V) modes recorded in Fig. 16 show amplitudes which were only noted on two occasions. During all other attempts the maximum steady amplitude indicated a considerably lower value. In general, the higher vertical modes could be built to maximum amplitude with little difficulty.

In Fig. 18(*d*), the motion of the model is in full control of the flow, and the maximum double amplitude has reached 1.4 in. (70 in. on the prototype). In the same run and at the same velocity a few seconds later, the amplitude has fallen off as shown in Figs. 18(*e*) and 18(*f*), and the record indicates that the control of the flow by the section is not complete—that is, some excitation on the subharmonic level is present again.

Only on the basis of the subharmonic theory of excitation can the double appearance of each of these modes be explained. There is some evidence, not yet well substantiated, that the first appearance is harmonically excited, whereas in the second appearance the excitation is subharmonic.

Part 4.—The author's general statements under the heading, "Anticipating Aerodynamic Instability," are not in full accord with experimental evidence derived from the full model which was tested with many variations of the configuration of the suspended structure. These configurations are listed in Table 8.

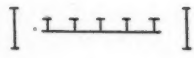
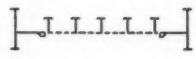
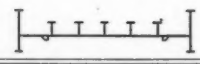
Lift, drag, and moment tests on fixed sections in the wind tunnel are available for all these configurations with the exception of sections 4, 5, 9, and 11, Table 8. It is significant to note that only in section 6 did a negative slope develop on the lift graph. Likewise, all these sections showed both vertical and torsional motions with the exception of section 10 in which no torsional motion developed in velocities up to a value greater than 110 miles per hr (prototype velocity). With great difficulty and only after very long runs, slight motion was observed in the two-noded and three-noded vertical modes in section 10. The maximum amplitude developed at the end of approximately 30 min (212 min, prototype) was 0.14-in. double amplitude in the two-noded mode.

Section 2 developed the most vigorous motion (see Fig. 16) in both vertical and torsional modes, and it is significant to note that the slope of its lift graph was strongly positive at all angles and that its moment graph was negative in slope over the middle range. All tests with girders of any depth showed negative slopes on the torque graph, and the only section that showed torsional stability (section 10) had a positive slope on the torque graph. In respect to

the torque graph for girder bridges, the laboratory tests are in complete agreement with the author's claims.

However, recent tests on section models and full models of a new design using a truss for stiffening show a positive slope on the torque graph between

TABLE 8.—COMPARISON OF GIRDER SECTIONS

No.	Section	Description
1		The section of the bridge as built
2		Roadway slab completely removed leaving stringers and solid sidewalks with curbs
3		Section 2 with opening covered by screen of proper mesh to simulate a conventional open deck
4		Solid roadway slab with sidewalks and curbs removed
5		Both sidewalks and roadway slab removed, stringers still in place
6		Girder deepened to equivalent of 13 ft with deck open as in section 2
7		Full circular fairing on both girders, open deck as in section 2
8		Full circular fairing on both girders, closed deck as in section 1
9		Section 7 with bottom enclosed
10		Section 7 with top and bottom enclosed
11		Section 10 with circular fairing removed

the $+12°$ and the $-12°$ angles of attack; yet the model exhibits strong torsional instability when the angle of attack is approximately $+7°$ or greater and a weaker manifestation of torsional instability when the angle of attack is $-7°$ or greater. This positive slope on the torque graph and resultant instability

of the section were likewise demonstrated in fixed model and section model tests at California Institute of Technology (unpublished report).

Use of Center Stays.—Center stays are only effective in inhibiting asymmetric torsional modes, and although they have been installed on a number of bridges it was never with this function in mind. They were first conceived as a means of reducing the longitudinal pendulum-like motion that is observed in connection with asymmetric vertical modes, and there is no evidence that they have ever been designed with regard to the stress that might arise in either the vertical or the torsional modes. There is some evidence to indicate that the center stays on the Tacoma Bridge failed (or yielded) in a strong eight-noded vertical motion which was active for some time before the transition to the first asymmetric mode in torsion. It is possible that this yielding during vertical motion permitted the development of a torsion with its accompanying much higher loading on the stays, which caused further yielding and ultimate failure (possibly accompanied by failure in the lateral system at midspan), leaving the structure quite unprotected against full development of catastrophic torsion.

Open-Floor Construction.—The writer is not so optimistic as the author regarding the beneficial effect of open-floor construction. Tests on the full model have been confined to the sections listed in Table 8, but it is significant that Messrs. Prandtl and Tietjens[47] showed a strong three-noded vertical mode with an amplitude as great as that shown in tests by Messrs. Theodorsen and Garrick[46] (the lower modes were very weak) and with a single-noded torsional motion in excess of 15° of amplitude. A much higher velocity was required to bring in each of these modes but their amplitude was disquieting.

It was confidently expected that the complete removal of sidewalks and roadway slab (section 5, Table 8) would vastly improve the stability condition; but tests involving two complete runs showed three-noded and four-noded vertical modes of large amplitude and also developed a three-noded torsional mode of strong character. The three-noded torsion was especially violent and reached an amplitude in excess of 15°.

Conclusions.—The writer is in complete agreement with the author's concluding statement that the most scientific approach to this problem calls for the elimination of the cause of the motion.